CCNETWORK Comprehensive Center Network

REGION 8

Indiana | Michigan | Ohio

Michael Stevens

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Brief

Geometry Supports and Resources for Teachers

CCNETWORK Comprehensive Center Network

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Mike Stevens is a math teacher and expert consultant living in New York's Hudson Valley. He taught all levels of high school mathematics at the High School for Law and Public Service in Washington Heights, NY. He holds a B.A. in Philosophy from the University of Florida and a master's degree in education from the City College of New York. Mike has extensive experience facilitating workshops and providing coaching for teachers, coaches, and leaders across the country. He is an expert consultant with a focus on formative assessment, curricular development, unpacking of content and practice standards, and practical strategies for effective instruction. Mike established Math Concepts Inc. in 2012 to provide expert consulting for the Gates Foundation's Math Design Collaborative as well as to support professional learning at a district level. He has been collaborating with the Region 8 Comprehensive Center since 2020.

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Background Information

The Region 8 Comprehensive Center (CC) is one of 19 Regional CCs in the CC Network that provide high-quality, intensive capacity-building technical assistance to clients from state, regional, and local educational agencies and schools in Indiana, Michigan, and Ohio. Region 8 CC staff serve clients by helping to identify, implement, and sustain effective evidence-based programs, practices, and interventions that support improved educator and student outcomes. Through these capacity-building services, Region 8 CC staff help agency staff improve educational outcomes for all students, close achievement gaps, and improve the quality of instruction.

This document contains a summary of the resources and supports that Michael Stevens has prepared addressing Geometry courses for the Region 8 Comprehensive Center.

Purpose

The purpose of this resource is to help math teachers unpack, understand, and implement the current math content and practice standards. This resource describes the progressions of learning within each course and provides content support, which includes broad ideas about effective instruction as well as practical instructional strategies. Math teachers, coaches, and leaders are encouraged to use these materials collaboratively to support ongoing instruction and the growth of individual teaching practice.

The content is organized by the following domains in Geometry, including:

- Standards for Mathematical Practice
- Geometry: Logic and Proofs
- Geometry: Points, Lines, and Angles
- Geometry: Transformations
- Geometry: Triangles
- Geometry: Quadrilaterals and Other Polygons
- Geometry: Circles
- Geometry: Three-Dimensional Figures

Each section includes a variety of Content Supports. There are also sections that focus on students who struggle and language and communication.

There is also a list of references and resources.

Standards for Mathematical Practice

The Standards for Mathematical Practice describe necessary academic and cognitive habits that help students access and understand the mathematical content at each grade level. The practices are based on proficiencies articulated in the National Council of Teachers of Mathematics (NCTM, 2000) process standards and the National Research Council's *Adding It Up* (2001).

While the content standards change for each grade and course, the eight Standards for Mathematical Practice are applied to all grade levels and intended to be developed continuously, year by year, as students progress from kindergarten through high school.

Standards for Mathematical Practice (MP)	
MP.1: Make sense of problems and persevere in solving them.	Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.
MP.2: Reason abstractly and quantitatively.	Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
MP.3: Construct viable arguments and critique	Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning

Standards for Mathematical Practice (MP)	
MP.7: Look for and make use of structure.	Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well-remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.
MP.8: Look for and express regularity in repeated reasoning.	Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Geometry: Logic and Proofs

Overview and Progressions of Learning

The National Council of Teachers of Mathematics (NCTM) identifies Reasoning and Proof as one of the five process standards essential for all math learners.

Elementary and Middle School

As students move through the elementary grades, they learn to see mathematics as a coherent logical system in which truth can be ascertained and communicated through justification and evidence. Reasoning and informal proof are developed through middle school as students are asked to defend their findings and justify concepts like equivalent expressions, valid solutions to equations, and the Pythagorean Theorem.

Students also develop their geometric understanding through the early grades,. By fourth grade they work explicitly with circles, angles, parallel and perpendicular lines, and line segments. Concepts of measurement in two and three dimensional space are developed over several grades as students solve problems about area and volume. In middle school students work with nets, transformations, congruence, similarity, and theorems.

The focus on logic and proof both relies on and helps develop several of the Standards for Mathematical Practice, including:

- *Reason Abstractly and Quantitatively* (MP.2)
- *Construct Viable Arguments and Critique the Reasoning of Others* (MP.3)
- *Look for and Make Use of Structure* (MP.7)
- *Look for and Express Regularity in Repeated Reasoning* (MP.8)

Students develop and hone these practices year by year, through conjecture, argumentation, and justification.

High School

In high school, the expectation for more formal explanation increases, and students extend their geometric learning from elementary and middle school to include precise definitions and formal proof. They understand valid assumptions and reason from premises to conclusions as they produce paragraphs, flowcharts, and two-column proofs. The practice of proof continues in later high school and college courses as students create both geometric and algebraic proofs that may be direct or indirect.

Content Supports

Listed below are five strategies to build students' conceptual understanding and procedural fluency related to *Logic and Proof*. The five strategies include:

- Real-World Scenarios
- Error Analysis
- Planned Debate
- Vertical Planning
- Daily Formative Assessment

Real-World Scenarios

Students use reasoning every day as they interact with the world and navigate various real-world problems. They consider available evidence, observable patterns, and prevailing logic.[MP.2,7,8] There are standards in every geometry domain that call for different levels of proof ranging from informal argument and demonstration to formal proof.

Ultimately the goal is that students use appropriate geometric definitions and relationships to create formal proofs. The use of real-world scenarios and engaging logic problems that activate students' current reasoning can build student confidence and a positive disposition towards geometric proof.

Real-world scenarios can be considered through riddles, brain teasers, conundrums, stories with "holes," logic games, puzzles, and picture problems. Such activities engage inherent student reasoning, encourage risk taking, and develop a classroom environment based on discourse, argumentation, equity, and justification.

When presenting riddles, it is important to give students private think time before discussion, encourage active listening, and communicate the expectation for thinking, sharing, and justification.

Below are some examples of riddles that support mathematical thinking and can be presented to students:

- *When my father was 31, I was 8. Now he is twice as old as me. How old am I?*
- *Mr. and Mrs. King have six daughters and each daughter has one brother. How many people are in the King family?*
- *I am not alive, but I grow; I don't have lungs, but I need air; I don't have a mouth, but water kills me. What am I?*
- *A man is looking at a picture of a man and states, "Brothers and sisters I have none, but this man's father is my father's son." Who is the man in the picture in relation to the man looking at the picture?*

Conundrums, or stories with holes, are stories about events that give some information and leave other information missing. They often push the listener to question the meanings in the story. After students are given time to think on their own, they can be invited to ask yes or no questions about the story or share conjectures to be tested by other students.

Below are examples of conundrums:

- *The man was afraid to go home because the man with a mask was there waiting for him. Who was the man? Who was the man with the mask? (Answer: The man with a mask was the catcher of the other baseball team. The man was a base runner trying to go home and score for his team.)*
- *A doctor was driving his son to school one day when their car was rammed by a truck. The doctor was killed and his son seriously injured. The son was rushed to a nearby hospital and prepared for surgery, however, when he was wheeled into surgery the surgeon announced: "I can't operate, this is my son." How can this be? (Answer: the surgeon was the boy's mother.*
- *John and Mary are on the floor. There are pieces of broken glass and a puddle of liquid on the floor. Mary is dead. What happened? (Answer: John, a cat, knocked over a fishbowl in which Mary, a fish, lived.)*

Again, the main value in these riddles is that students take part in the discussion, share their reasoning, listen to others, and gain confidence in their thinking. It is important for them to know they are allowed to be wrong, to not know, to make mistakes, to ask questions, and to change their minds.MP.1 Also, these type of stories often have other plausible and reasonable explanations, which should be honored, but the goal is to get students to reason toward the actual explanation through questioning and flexible thinking.

Real-world scenarios can also be used as students learn about different types of logical statements. As students learn about different logical statements, they can be asked to apply these logical structures to real-world scenarios and also invited to create their own applications.

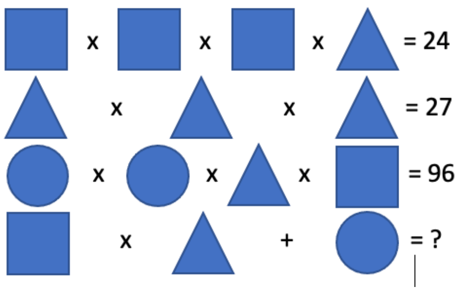
> For example, the conditional statement "*If I didn't get enough sleep, then I am tired*" may be seen as true, but the converse "*If I am tired, then I did not get enough sleep*" may not be true. It is important that students are able to discuss, argue, and justify real-life statements that may be seen in different ways.

Other conditionals like, "*If a quadrilateral is a square, then it is a rectangle,*" have a more clearly defined truth value, as do their converse, inverse, and contrapositive, but students should still be encouraged to justify their evaluations and construct arguments that can be used to convince others. More broadly, students can be asked to form proofs about real-life events which reason from premises to conclusions and are convincing to the careful listener.

Picture puzzles can also be useful to elicit student reasoning, argument, and proof. They engage visual learners, are generally non-routine, and tend to be accessible to all students.

Below is an example:

The three glasses on the left are full, while the three on the right are empty. Make a row of alternately full and empty glasses by moving only one glass.



When solving picture puzzles, discourse expectations including private think time and rules for calling out should be made clear. Students should be given tools such as white boards to experiment, record, and communicate their thinking, and all findings should be fully justified.

Once answers are justified and agree upon, students can be asked to share their processes, starting points, key sense making, or mistakes.[MP.1,5]

Error Analysis

The ability to see mistakes as opportunities for learning is an essential part of a developed math practice. A positive attitude toward identifying and correcting mistakes helps students learn to make sense of problems and persevere in solving them.

According to MP.1, proficient students:

- Monitor and evaluate their progress and change course if necessary
- Check their answers to problems using a different method,
- Continually ask themselves, “Does this make sense?” and "Is my answer reasonable?"

Error analysis is a practice that helps develop these abilities by making mistakes the central focus of the learning and giving students the opportunity to identify the mistakes, correct them, and justify the corrections. Errors are often presented as the work of a mystery student or a student in another class, which allows students who are uncomfortable with their own mistakes to engage in the math without feeling defensive.

It is also essential that the mistakes presented are not simply careless errors, but stem from common misconceptions specific to the content and apply to the thinking of a significant number of students. This requires thoughtful formative assessment of ongoing student work. Mistakes that could apply to L*ogic and Proof* standards include:

- Inappropriate counterexamples in an argument,
- Neglecting to use given statements or justify steps in a proof, wrong application of key definitions,
- Misapplication of converse, inverse, and contrapositive, or
- Incorrect assessment of logical statements.

Have students work in pairs or groups. Present them with cards that show geometric work related to logic and proof that includes at least one mistake. The students' task is to identify the mistake, describe the misconception, and then complete the problem correctly.

This type of activity generates rich student discussion and allows students to take on the role of teacher as they explain and correct the mistakes. When students seem to grasp the most common mistakes in the math content, they can also be asked to create their own problems with mistakes and present them to other pairs or groups for correction. As students work with common errors around specific content, they will be more likely to see the misconceptions that drive common errors and to anticipate them and avoid them in their own work.

Planned Debate

Planned Debate is a discourse strategy that supports logic and reasoning by encouraging students to make inferences, gather evidence, and justify conclusions. The debate can be arranged such that students pick the side they wish to argue, or students could be randomly selected to argue a particular side in the debate.

The debate can be arranged in pairs, small groups, or with the whole class and can be public or private. The focus of the debate can be more subjective, such as whether triangles are better than quadrilaterals or vice versa, where students argue different sides and are tasked to be as convincing as possible.

The debate can also be about a situation where the truth value or outcome is clearer (whether two walls are parallel, whether two shapes or angles are congruent, whether a property of a shape is always true, etc.), and each team's job is to construct and deliver the most convincing argument.[MP.3]

A similar structure that could be used is called silent debate, where students are asked to debate a question silently by writing back and forth on one sheet of paper with a partner. Both of these strategies build justification and reasoning, support listening skills, and engage students in the vital sense-making indicated in the *Logic and Proof* standards.[MP.3] Debates also create a place in which sharing your thinking, working to justify your reasoning, and making convincing arguments become highly valued practices.

Vertical Planning

The teaching and learning in every domain in the Geometry course needs to be planned and implemented with attention to connections between standards in different domains. Vertical planning can be thought of as "stacking'" standards from different domains or teaching more than one standard at a time.

Standards that include logic and proof support various kinds of logical reasoning and more formal proof applied to content in several domains, including *Congruence*, *Similarity and Right Triangles, Circles*, and *Expressing Geometric Properties with Equations*.

In the *Congruence* domain, standards CO.9-11 ask students to prove a number of theorems that they will use later to solve problems in several other domains. Many of the basic terms, notation, and relationships addressed in these theorems are first addressed by definition in standard CO.1, and the structure and practice of proof is also addressed in standards CO.7-8, SRT.3-5, C.1 and GPE.5. These related standards could be addressed with an intentional sequence based on learning goals, and they could also be taught together as students practice constructing proofs relating to various geometric objects.

There are many ways to attend to vertical planning. Connections between standards in different domains should be taken into consideration during curricular design and sequencing. In some cases, it may be clear that one standard should be addressed prior to another, but in other cases it may be more a matter of noting the resonance between standards. Effective vertical planning helps teachers make better use of instructional time overall, reduce the time between instruction of related content, and increase time spent on high priority content.

Regardless of how related domains are sequenced within the curriculum, these connections should be noted and intentionally addressed within planned lessons and activities. If teachers are intentional about vertical planning, they will gather and respond to assessment with the later standards in mind and will naturally refer back to previously addressed standards when focusing on standards that come later. They will also be more likely to keep language, representations, and procedures consistent and build more effectively on prior learning.

Daily Formative Assessment

Students have been developing their reasoning and argumentation skills since the early grades, and in high school they begin to formalize those practices into proofs that address specific geometric learning. The practice of proof and justification is essential to all geometric thinking and impacts learning in every domain in the course.

Broad assessment questions like "What is a proof?"; "Why is logic an important part of mathematics?"; and "Where do you use logic in your daily life?" may be essential questions for a student or group of students as they address these standards. Essential questions should be intentionally planned and asked repeatedly throughout the learning. Students benefit from answering from their own thinking and refining their expression over time, and teachers benefit from the subtle anecdotal assessment they receive.

This type of informal formative assessment shapes ongoing instruction. Teachers who include open conceptual questions as part of daily lesson plans develop more effective questioning practice overall.

Daily formative assessment may be formal or informal. It is often anecdotal and generated from classroom discussions with individual students or groups of students. Formative assessment is based on asking good questions. Assessment data can be gained through checks for understanding where students respond in the moment either verbally or physically (raising hands, or giving a thumbs up or thumbs down). It can also be generated through exit tickets or reflection questions given at the end of a class or at the completion of an activity.

In all cases, formative assessment is conducted while students are still learning the material. Assessment should generate data about *how* they are learning the material and what misconceptions are arising, so that teachers have time to adjust instruction and students have additional opportunities to build understanding.

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are “Why?” and “What are they missing?” Only through consistent discourse and formative assessment can we begin to know why they are not mastering the standard. What is it that they do not understand? And how far back do their misconceptions go in the prerequisite standards?

If we can identify the missing concepts, we can begin to understand why students are struggling with the current standard. Students may also begin to grasp some of the conceptual parts of the standards and yet struggle with specific language or procedures.

For students who struggle with logic and proof, it is important that students connect their own inherent reasoning that they use in everyday life to the expectations of the current standards. Students that see geometric proof and logical reasoning as something they add on to their own thinking and use only for math class are less likely to retain significant understanding over time.

Students need to understand basic definitions and terms fully, and exhibit ownership of that understanding. This can be done through intentional discussion, questioning, expectation for justification and explanation, and the practice of having students draw and justify basic geometric objects.[MP.6] All of these practices serve to confirm understanding, to share student understanding among peers, and to surface misconceptions that can be used to inform later instruction.

In addition to the broad formative assessment questions noted above, the following questions can be helpful in assessing misconceptions and developing understanding and procedural fluency related to proof and logic.

- What is a proof and why are they important?
- What is a counterexample and why are they useful in an argument?
- What are some geometric axioms or postulates that you have learned that might be useful in a proof?
- What is an angle? What kind of space does an angle measure?
- What is a circle and what are some important features in a circle?
- What are some of the differences between parallel and perpendicular lines?
- Can one line be perpendicular? Why or why not?
- What are the main differences between a point, a line, and a plane?
- What dimensions do points, lines, and planes each represent?
- What is a conditional statement? What determines their truth value?
- What are the essential parts of a proof?
- What kind of proof do you prefer and why?
- What is the hardest thing about writing a proof?

The formative assessment of logic and proof should be ongoing through most of the other domains in the course. These and other assessment questions can be used repeatedly throughout the course as students continue to develop their practice and understanding.

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including the following:

- Revoicing – "So you're saying that…."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition.

Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work, and actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

Encourage Student Sense-making

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition.

The best questions to encourage sense-making and justification are "Why?"; "What does that mean?"; and "How do you know?" These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment data. Also, they can be asked every day. As students justify the steps in their reasoning and how those steps relate to accepted geometric ideas, whether engaging in informal argument or more formal proof, the role of classroom discussion in establishing and assessing student thinking could not be more important.

Basic geometric terms and definitions will be used throughout the course and should be re-defined and justified by students as much as possible. It is common for students to know something about a key term and yet also have misconceptions about it, and by continuing to ask them what different terms mean we are more likely to uncover such misconceptions. Logic and proof require a high level of student centered discourse in order to assess and teach them effectively. The important terms addressed and the process of going from informal reasoning to proof require students to talk and express themselves regularly as well as consider and critique the reasoning of others.

Vocabulary

As noted above, key vocabulary terms need to be discussed and defined by students again and again, allowing them to build and refine their understanding of each concept and the connections between them.

- Axiom
- Postulate
- Proof
- Undefined Terms
- Theorem
- Counterexample

- Angle
- Circle
- Perpendicular
- Parallel
- Point
- Line
- Plane
- Infinite
- Dimension
- Truth value
- Premises
- Given
- Hypotheses
- Conjecture
- Conclusion
- Two Column Proof
- Paragraph Proof
- Flow Chart Proof

Geometry: Points, Lines, and Angles

Overview and Progressions of Learning

Points, lines, and angles are important geometric figures that form a foundation for plane geometry. Essential geometric ideas like congruence and basic theorems that inform later work with polygons and problem-solving begin with these basic figures. These include angle pair relationships, parallel and perpendicular line relationships, and the use of the Pythagorean Theorem to find distance on a coordinate plane. All these topics have been addressed in prior grades, and students are expected to access, refine, formalize, and extend prior learning to meet geometry standards in high school.

Elementary and Middle School

In fourth grade, students define and classify angles, measure angles by degree, and classify shapes by the nature of their angles. In fifth grade, students classify two-dimensional figures by properties including angles. In seventh grade, students apply properties of pairs of angles (vertical, adjacent, complementary, and supplementary) to solve real-world problems.

The topic of parallel and perpendicular lines is also addressed in fourth grade and continues through the study of parallelograms and other quadrilaterals in middle school In eighth grade, students learn that rigid motion transformations preserve parallelism.

High School

In Algebra 1 students investigate the slope criteria for parallel and perpendicular lines. Also in eighth grade, students learn about the idea of congruence through applications of rigid motion and apply the Pythagorean Theorem to solve problems and measure distance.

All of these ideas are essential for a full understanding of geometry. Previous experiences should be honored and utilized while student understanding is assessed and extended. Many of these ideas become key understandings within the Geometry course and play key roles in later courses on Euclidean and non-Euclidean geometry.

Content Supports

Listed below are five strategies to build students' conceptual understanding and procedural fluency relating to *Points, Lines, and Angles* domain. The five strategies include:

- Real-World Scenarios
- Flexible Procedures
- Vertical Planning
- Multiple Representations and Tools
- Daily Formative Assessment

Real-World Scenarios

Students benefit from applying all mathematical learning to real-world problems.[MP.4] Application builds a value for the mathematics, deepens conceptual understanding, and allows students to develop their quantitative and abstract reasoning practice.[MP.2] Geometric learning about line and angle relationships and measurements describe the real space we live in, so there are countless real-world scenarios that could be modeled by this math.[MP.4]

A common and productive application for this content is civil planning including the design of streets, intersections, parks, and other municipal structures. Vertical angles and angles created by parallel lines cut by a transversal are easily included in such applications, as are parallel and perpendicular lines, perpendicular bisectors, slope, and distance. These figures can be given in a diagram, the diagram can be created by students, or students can also be asked to construct the figures using hands-on or virtual tools.[MP.5] These topics can also be applied to problems about architecture, design, tile patterns, and tessellations. Problems can be presented in two or three dimensional space. Real-world applications such as these engage students' prior experiences in real space and can help them think more deeply about geometric relationships in real space as their understanding develops.

Concepts, such as proving the slope criteria for parallel and perpendicular lines and finding the area and perimeter of polygons on the coordinate plane, can be engaged through application contexts that include movement and can be examined on and off the coordinate plane. A figure moving a number of units east and a number of units north creates a slope that can be calculated and compared with other linear movements. Distances related to moving objects as discrete values or as a function of time can be calculated using the Pythagorean theorem, on or off the coordinate plane.

Another way to utilize real-world scenarios is through direct student experience. Concrete experiences help students make sense of angle and line relationships by activating their spatial reasoning. For example, if two parallel lines cut by a transversal are modeled with tape on the floor, students can occupy different angle locations, discuss relationships, and justify congruence or other relationships.

Concrete experiences can also be used to investigate slopes in general and can include movement activities. Students can model lines with parallel and perpendicular slopes while moving on a coordinate plane marked on the floor to get a better sense of why specific slope relationships create certain line relationships. This particularly helps students internalize that perpendicular slopes are not just reciprocals, but must have opposite signs as well. The subjective perspective of actually being in the space can really help students who have difficulty conceptualizing spatial relationships abstractly.

Flexible Procedures

Procedural fluency is an important part of a student's math practice, and flexibility is a significant part of a student's procedural fluency. As students work with points, lines, and angles, encouraging the flexible use of procedures is essential. This includes allowing students to experiment, giving them choice in tools and method, and allowing them to make and correct their own mistakes.[MP.1] Flexible procedures are supported not when content is simplified into a set of steps for students to follow, but rather when students are asked to choose and justify their procedures.

Consider the following standard from the *Congruence* domain.

> CO.9 *Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

As students investigate and identify these angle relationships, there are often multiple ways to justify relationships including vertical angles, supplementary angles, and other theorems. Allowing students to choose, define, and justify their pathways helps develop a well-rounded math practice, and can be done individually, in pairs, in groups, or with a teacher. Different students will move toward higher levels of efficiency at different rates, but they will have a deeper understanding of that efficiency once it is acquired.[MP.7-8]

While it is true that the slope of a line is generally described as the change in y over the change in x, there are many ways to find the slope of any graphed line. Students should be encouraged to make their own choices about how to do so. These include the choice of which two points to consider, which point to start with, whether to consider horizontal or vertical change first, how to include any negative movement, and how to represent and validate their findings. Student choices should be driven by their understanding of what they are trying to achieve, and they should be expected to justify their choices in light of those goals.[MP.1]

As students learn to use the Pythagorean Theorem to calculate distance, they can be given choice as to which triangles are used to calculate distance and how to arrange any equations they use. The distance formula is simply a manipulation of the Pythagorean Theorem with coordinate differences in place of leg lengths. Many students can learn to reliably find distance between two points or the lengths of line segments by flexibly applying the Pythagorean Theorem and never memorize the distance formula in its traditional form.[MP.7] Flexible procedures allow students to apply their own conceptual understanding.

The geometry standards require that students "make formal geometric constructions with a variety of tools and methods."[CO.12] This practice emphasizes the foundational importance of points, lines and angles because the classic tools of straightedge and compass create all constructions from these three figures.[MP.5] Constructions are less about students following or memorizing a prescribed set of steps and more about students understanding the meaning of the process and purpose of each step. Teachers support flexible procedures when they allow students to explore and try different things during constructions, make conjectures about their next steps, and then verbally reflect on the outcomes. This includes verbalizing their understanding about the meaning of the figure to be constructed and then considering the necessary structures and relationships that would determine such a figure.

At the heart of a value for flexible procedures is not only the appreciation of the role that procedural fluency plays in mathematical learning, but also the insistence that instructional time is focused on student thinking and student sense-making as much as possible. This takes time, requires planning, and relies on strong relationships with students, but it is also integral to students increasing ownership of their learning and developing a well-rounded math practice.

Vertical Planning

The teaching and learning in every domain in the Geometry course needs to be planned and implemented with attention to connections between standards across domains. Vertical planning can be thought of as "stacking" standards from different domains or teaching more than one standard at a time. Several standards related to *points, lines, and angles* are supported by or provide support for standards in other domains or adjacent clusters.

Students have worked with the geometric objects noted in standards CO.9 and GPE.5 during prior courses, and their definitions and notations are also called for in standard CO.1. The understanding of proof required to prove the theorems noted in CO.9 is also addressed in standards GPE.4-5. These standards could be addressed sequentially in separate units (with explicit reference to connections between standards across domains); but the content in these standards is so interdependent, that they could also be addressed within one unit of learning.

The basic angle relationships noted in standard CO.9 and the line relationships found in GPE.5 form the basis for the investigation of many other geometric topics, including theorems and proofs about triangles and other polygons. Several theorems mentioned in standards CO.10 and SRT.4 are proven through line and angle relationships, particularly the angle relationships created by two lines cut by a transversal. Similarly, proofs and definitions of quadrilaterals as noted in standard CO.11 rely heavily on angle relationship and concepts about parallelism addressed in standard CO.9. While these standards may be addressed in different units of learning, attention to vertical planning means that connections between such standards is considered by teachers and students and the learning destinations implied by standards is discussed.

There are many ways to attend to vertical planning, but connections between standards in different domains and clusters should be taken into consideration during curricular design and sequencing, lesson planning, and lesson delivery. Effective vertical planning helps teachers make better use of instructional time overall and can reduce the time between instruction of related content. Regardless of how related domains are sequenced within the curriculum, these connections should be noted and intentionally addressed within planned lessons and activities. If teachers are intentional about vertical planning, they will gather and respond to assessment with the later standards in mind and will naturally refer back to previously addressed standards when focusing on standards that come later. They will also be more likely to keep language, representations, and procedures consistent and build more effectively on prior learning.

Multiple Representations and Tools

As in most geometric learning, the use of multiple representations and visual learning is essential for students as they address the standards relating to *points, lines, and angles*. Individual white boards are a powerful tool for student experimentation and important formative assessment.[MP.5] They are low investment for students and easily erased,. White boards also provide a semi-permanent record of student thinking that can be communicated, justified, or questioned.[MP.3]

Student-created diagrams help build conceptual understanding not only of the relationships and theorems indicated by the standards but also of the given conditions that make them possible. White boards can be used to apply angle or line relationships to a given diagram. But asking students to create a representation of given conditions from words also helps bring common misconceptions to the surface. For example, some students can identify angle relationships when given a diagram of parallel lines cut by a transversal. But when students are asked to draw this structure, confusion is revealed about which lines are parallel, which line is the transversal, how the figure can be varied, and how the figure should look. This practice can also reveal which students struggle to conceptualize geometric relationships when the given figure is presented with a different orientation.

In standard CO.12, students are expected to create and explain geometric constructions of various objects that they have been exposed to in prior courses. The tools involved, compass and straightedge, may be new to many students and can be difficult to use.[MP.5]

Students need significant practice, instruction, and the ability to help and teach others how to use the tools in order to gain proficiency so that they can focus on the geometry. Students also benefit from being asked to consider what it means that all constructions can be created with only these two tools.[MP.7,8] The power of a circle or arc as a measurement of distance is an essential geometric idea that gives meaning to the practice of constructions.

Virtual geometry software, such as *Geogebra,* can help students understand constructions more quickly and allow constructions to become an investigation and justification tool for students rather than just a one-time event where they create different objects.

Another important representation to consider is concrete experience, which can help students make sense of angle and line relationships by activating their spatial reasoning. For example, having students occupy different kinds of angles in real space based on figures taped to the floor or marked with rope allows them to experience the geometry in a different way. Students could create life size constructions using a human compass, investigate slope relationships, or discuss and justify angle relationships. The subjective perspective of actually being in the space can really help students who have difficulty conceptualizing spatial relationships abstractly.[MP.2] As with all representations, not all students may need concrete experiences to understand the geometry, but experiencing geometric relationships in real space and seeing the figures from within builds student conceptual and spatial understanding that some students may not acquire through other presentations.

Daily Formative Assessment

As students apply geometric understandings from prior courses and seek to gain fluency with the theorems, objects, and relationships related to points, line, and angles, the practice of daily formative assessment is essential. Intentional formative assessment questions and tasks help students reflect on their own learning and inform teachers what skills and concepts are being transferred from prior learning.

Broad assessment questions like the following may be essential questions for a student or group of students at different times:

- "What is a theorem?"
- "What are parallel and perpendicular lines?"
- "Why do we use only a compass and straight edge for constructions?",
- "What is the Pythagorean Theorem used for?"

Essential questions should be intentionally planned and asked repeatedly throughout the learning. Students benefit from answering from their own thinking and refining their expression over time, and teachers benefit from the subtle anecdotal assessment they receive. This type of informal formative assessment shapes ongoing instruction. Teachers who include open conceptual questions as part of daily lesson plans develop more effective questioning practice overall.

Daily formative assessment may be formal or informal. It is often anecdotal and generated from classroom discussions with individual students or groups of students. Formative assessment is based on asking good questions, and can be gained through checks for understanding where students respond in the moment either verbally or physically (raising hands, or giving a thumbs up or thumbs down). It can also be generated through exit tickets or reflection questions given at the end of class or at the completion of an activity.

In all cases, formative assessment is conducted while students are still learning the material. Assessment should generate data about *how* students are learning the material and what misconceptions are arising so that teachers have time to adjust instruction and students have additional opportunities to build understanding.

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not mastering the standard. What is it that they do not understand? And how far back do their misconceptions go in the prerequisite standards?

If we can identify the missing concepts, we can begin to understand why they are struggling with the current standard. Students may also begin to grasp some of the conceptual parts of the standards and yet struggle with symbolic representations and notation.

For students who struggle with learning related to points, lines, and angles, it is important to identify how they think about the different geometric objects and relationships addressed in the standards. This can be done through intentional classroom discourse, student created drawings and diagrams, and the use of formative assessment tasks and activities. Students should be able not only to identify objects like angles, parallel lines, and bisectors, but also to explain the different possible variations within each class of objects.

Misconceptions about key concepts or relationships create difficulties for students if left undiscovered, but if brought to the surface they can lead to deeper connections and allow students to build their understanding as well as their mathematical practice.

In addition to the broad formative assessment questions noted above, the following questions can be helpful in assessing misconceptions and developing understanding and procedural fluency.

- What is a theorem and why are they important in geometry?
- What do all angles have in common?
- How do we measure angles?
- What are some different kinds of angles?
- Are angles one-dimensional or two-dimensional?
- What are some important angle pair relationships?
- What makes two angles congruent?
- Why are vertical angles congruent?
- What do two congruent angles have in common? In what ways can two congruent angles be different?
- What is a transversal?
- What does a bisector do?
- What is slope and how do you find it?
- Why do parallel lines have the same slope?
- Why aren't two lines with opposite signed slopes perpendicular?
- What is the difference between perpendicular lines and intersecting lines?
- When making constructions, which tool is used to measure distance?

- How do you construct a perpendicular bisector? Which parts of the construction make it perpendicular? Which parts make it a bisector?
- How is the construction of a perpendicular bisector different from the construction of a perpendicular line?
- What does the Pythagorean Theorem say?
- How can the Pythagorean Theorem be used to find the distance between two points?
- Can the Pythagorean Theorem be used to find the distance between ANY two points? How?

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including the following:

- Revoicing – "So you're saying that…."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words.

The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work. Acquisition actually develops over time, if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

Encourage Student Sense-making

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition.

The best questions to encourage sense-making and justification are

- "Why?"
- "What does that mean?"
- "How do you know?"

These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessments. Also, they can be asked every day.

In addition to the expectation that students use appropriate vocabulary, they should also be given repeated opportunities to define that vocabulary over time.[MP.6] Because students are familiar with many of the terms in these standards, teachers may assume that they fully understand each term, but upon further discussion may find that this is not the case. Students may also have subtle misconceptions that hide behind real understanding and need to be teased out, such as the idea that all bisectors are perpendicular or that all parallel lines are horizontal.

Language around angles and their relationships should be particularly intentional. It is important that students conceptualize and differentiate between language that relates to individual angles (vertex, ray, measure, acute, etc.) and language that relates to angle pair relationships (congruent, vertical, supplementary etc.).[MP.6]

When making constructions, opportunities for classroom discourse should be intentionally designed. This can include discussion questions or reflections as students work in groups, or prepared, open questions for a whole class discussion. Students should be expected to verbalize and justify the moves in their constructions, their goals with each move, and any mistakes that are made.

Vocabulary

As noted above, key vocabulary terms need to be discussed and defined by students again and again, allowing them to build and refine their understanding of each concept and the connections between them.

- Point
- Angle
- Angle Bisector

- Altitude
- Median
- Distance
- Pythagorean Theorem
- Midpoint
- Theorem
- Vertical Angles
- Alternate Interior Angles
- Alternate Exterior Angles
- Supplementary Angles
- Corresponding Angles
- Line Parallel
- Transversal
- Perpendicular
- Bisector
- Perpendicular Bisector
- Equidistant
- Slope
- Reciprocal
- Congruent
- Segment

Geometry: Transformations

Overview and Progressions of Learning

Elementary School

During elementary grades, students explore shapes by comparing, composing, and decomposing simple figures. They also identify the attributes and properties of basic shapes. In their algebraic work, students develop an understanding of equivalence and equality and a sense of why these are important mathematical ideas.

Middle School

In middle school, as students learn about ratio and proportional reasoning, they understand similarity as involving proportional relationships. Middle school students also create scale drawings and use ratio and proportion to solve problems about linear measurements. In eighth grade, students learn about congruence primarily through the application of rigid motion transformations. They also learn that if one shape can be mapped to another with perfect overlay by a series of rigid motions, then the two shapes are congruent.

High School

In high school Geometry, students continue to deepen their understanding of these essential geometric ideas. They broaden their understanding of congruence from rigid motion transformations to congruence within and between various kinds of figures to proving the congruence of polygons. Students extend their understanding of similarity as they engage with standards about right triangles, trigonometry, and circles. Transformations are a large part of that learning, and these essential ideas also impact standards and learning in all other domains in the course.

Content Supports

Listed below are five strategies to build students' conceptual understanding and procedural fluency related to *Transformations* domain. The five strategies include:

- Real-World Scenarios
- Open Tasks
- Congruence and Similarity
- Hands-On Activities
- Daily Formative Assessment

Real-World Scenarios

Standards CO.6-8 form a cluster within the *Congruence* domain entitled *Understand congruence in terms of rigid motions*. Standards SRT.1-3 comprise the first cluster of the *Similarity, Right Triangles, and Trigonometry* domain entitled *Understand similarity in terms of similarity transformations*. As students deepen their grasp of congruence and similarity through these standards, application tasks that include real-world scenarios continue to be essential in connecting mathematical experience to everyday life and building lasting conceptual understanding.[MP.4]

The standards call for an application of rigid motions described by translations, rotations, and reflections. These transformations are commonly applied within the geometry of everyday life, and connecting them to real-world scenarios activates students' experiential understanding. Translations can be used to model the layout for wall hangings, the rearranging of furniture, or the movement of soccer players on a field.

Reflections can be applied to experiences with mirrors of all kinds, the imitation of a dancer on a computer screen, or the imprint of a hand in clay. Rotations can be used to describe the spinning of a label on a DJ's record, the movement of hubcaps on a spinning wheel, or the shifting of the night sky around the pole star. What is most important is that the contexts help activate and extend student understanding of the nature of each rigid motion.

Real-world applications should add to students' experience of what defines and governs translations, rotations, and reflections, including how they are alike and different. Contexts also help students understand and verify the idea of congruence being maintained, effects on orientation, and the preservation of angles, parallelism, and distance.

The properties of dilations described in SRT.1 can be applied to points, lines, and polygons. Real-world applications include business logos, scale drawing, maps, and shadows. The idea that dilations are governed by a scale factor and involve proportional reasoning connects to prior learning from middle school, including multiplicative relationships and the constant of proportionality. Essential concepts for this content include the idea that images under dilation are not congruent to the pre-image but are similar, that proportional scaling affects all parts of the pre-image, and that scale factors less than one reduce the size of the image while those greater than one increase it.

Real-world scenarios support student learning to the extent that they help students recognize the geometry of transformations in everyday life and extend that understanding by investigating the properties of those transformations. Students should be asked to apply the geometry to given contexts and also to identify application contexts from their own experiences that could relate to transformations.

Open Tasks

One way to ensure that daily instruction includes conceptual learning and is not overly procedural is to include open or non-routine tasks and activities. Open tasks encourage creative thinking, require flexible reasoning, have multiple entry points, and cannot be accomplished solely through replicating procedures.[MP.1] They also often include multiple solution paths that could be justified.

Open tasks related to transformations could include giving students a graph of the pre-image and image of a figure (such as a polygon) and asking them to identify the transformation or combination of transformations that could apply. Students can also be asked to identify a single transformation that will map to the same location as a combination of transformations, or vice versa. It may be useful to include discussion or reflection questions along with open tasks in order to gain deeper assessment of student thinking. Questions about how students determine the presence of different transformations, effects on orientation, how to define movement, notation, or preservation of attributes could be useful.

Other useful open structures include "tell me all you know about…" in which students are asked to share prior or current knowledge around a mathematical topic. Questions about "how many ways…" can also be productive, as when students are given a set of graphs that each include the same shape in different locations and orientations and are asked to define the possible series of transformations involved. Activities like these foster creativity, offer choice as to where to start and what to include, and focus on reasoning and justification over having everyone generate the same outcome.

Congruence and Similarity

As students address these standards and learn to apply the procedures of the different transformations using appropriate notation, it is important to maintain a focus on the large themes of congruence and similarity. These are essential ideas about how geometric figures relate to one another as well as how objects move within two and three-dimensional space.

Transformations that describe rigid motions help students develop an understanding of and justification for congruence, because two figures can be seen as congruent when a series of rigid motions (reflections, rotations, or translations) map one figure onto the other with perfect layover. Rigid motions also preserve angle measure, parallelism, and distance, and only reflections reverse orientation. As students practice with rigid motion, questions about congruence and expectation for explanation helps to clarify the work, verify results, connect procedures to the underlying concepts, and extend conceptual understanding.[MP.8]

Learning about dilations activates important proportional reasoning and also supports the key concept of similarity. Asking questions about the concept of similarity while students practice the procedures of dilations can be equally productive to engage conceptual reasoning and understanding and give valuable assessment.

Intentional questions, tasks, and classroom discussions are essential to ensuring that students develop a deep understanding of the character of rigid and non-rigid transformations and connect these topics to the larger ideas of congruence and similarity. These ideas are priority content that relate to work in several other domains and are foundational to geometric understanding in the present course as well as higher courses that follow.

Hands-On Activities

Learning involving transformations provides rich opportunities for hands-on activities and work with manipulatives.[MP.5] Not only are these events engaging for students, but they are also critical for many students to actually understand the mathematics. While some students can conceptualize transformations using abstract reasoning, many benefit from or require concrete bodily experience of transformations in order to gain fluency.[MP.2] Imagining or describing a reflection or rotation is vastly different from experiencing it with a manipulative or the body itself and observing the results.

Students can be asked to apply transformations on real-world objects and observe the process and outcomes. They could be given a cut-out of a polygon or shape and practice transforming the shape on a coordinate plane. Groups could be challenged to find a transformation (or combination of transformations) that generate the image formed by another group, or students could be encouraged to experiment and define transformations that generate an image with a certain orientation. Finally, some students may benefit from "acting out" certain transformations or participating in them subjectively, such as rotating the body or acting out a mirror reflection.

All these activities include experimentation, are governed by the rules of real space, include choice, encourage discourse, and provide opportunities for collaborative work. Experiences with hands-on activities are a key strategy that supports conceptual learning and connects students' procedural learning with concrete experiences.

Daily Formative Assessment

A full understanding of transformations includes specific procedures and notation and also develops larger conceptual themes of learning. Effective formative assessment helps students connect procedures to those conceptual themes while increasing student understanding of both. This can be seen as the "why" that justifies the "how."

Broad assessment questions like "What is congruence?"; "How can we define congruence through rigid motions?"; and "How is similarity different from congruence?" may be essential questions for a student or group of students as they address these standards. Essential questions should be intentionally planned and asked repeatedly throughout the learning. Students benefit from answering from their own thinking and refining their expression over time, and teachers benefit from the subtle anecdotal assessment they receive. This type of informal formative assessment shapes ongoing instruction. Teachers who include open conceptual questions as part of daily lesson plans develop more effective questioning practice overall.

Daily formative assessment may be formal or informal. It is often anecdotal and generated from classroom discussions with individual students or groups of students. Formative assessment is based on asking good questions, and can be gained through checks for understanding where students respond in the moment either verbally or physically (raising hands, or giving a thumbs up or thumbs down). It can also be generated through exit tickets or reflection questions given at the end of class or at the completion of an activity.

In all cases, formative assessment is conducted while students are still learning the material. The assessment should generate data about *how* students are learning the material and what misconceptions are arising, so that teachers have time to adjust instruction and students have additional opportunities to build understanding.

Tasks and activities designed for formative assessment are also an essential part of daily formative assessment.

The following link shows an example of a formative assessment lesson from the Mathematics Assessment Project involving transformations.

- Transforming 2D Figures

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not mastering the standard. What is it that they do not understand? And how far back do their misconceptions go in the prerequisite standards?

If we can identify the missing concepts, we can begin to understand why they are struggling with the current standard. Students may also begin to grasp some of the conceptual parts of the standards and yet struggle with symbolic representations and notation.

For students who struggle with transformations, it is important to identify how they think about the procedures to perform transformations, the meaning of each transformation, and the concepts of congruence and similarity. Students study these concepts as well as the specific transformations in eighth grade, but their conceptual understanding and retention needs to be intentionally assessed and may still be developing.

The use of hands-on and open tasks and activities provide important opportunities for teachers to gain assessment through observation and listening to student discussions. Intentional questioning helps assess whether difficulties are procedural or conceptual in nature and can identify when students are trying to apply and memorize procedures without connecting them to meaning.

In addition to the broad formative assessment questions noted above, the following questions can be helpful in assessing misconceptions and developing understanding and procedural fluency related to transformations.

- What does it mean for two shapes to be congruent?
- How do rigid motion transformations show congruence?
- Why does orientation stay the same under translations and rotations but reverse under reflections?
- What information do you need to perform a translation?
- What information do you need to perform a reflection?
- What information do you need to perform a rotation?
- What aspects of a shape are preserved under rigid motions?
- What do we call the figure prior to transformation? What do we call the figure resulting from a transformation?
- What does it mean for two shapes to be similar?
- What do dilations have to do with similarity?
- What information do you need to perform a dilation?
- How does the scale factor value affect whether the dilation makes the figure bigger or smaller?

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including the following:

- Revoicing – "So you're saying that…."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words.

The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work. Acquisition actually develops over time, if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

Encourage Student Sense-making

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.MP.3 The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are "Why?"; "What does that mean?"; and "How do you know?" These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

Some language around transformations that needs to be attended to carefully is any talk about the pre-image and image. These terms are specific and important, as they help students understand that transformations are a process with a beginning and an end. It is also important to actively listen to how students talk about transformed figures; and, whether they communicate the idea that every part of the figure is transformed under the same rule as well as the whole figure and that the image is the result of the transformation being applied to the pre-image.

Finally, use of precise terms for each transformation is important and needs to be consistently upheld. Casual terms like flip, spin, or slide may help students recognize each transformation, but they should move quickly to precise vocabulary. The proper terms, including translation, rotation, reflection, and dilation, are also important for English language learners because they more closely resemble the equivalent words from many common languages and are more likely to be acquired with meaning.

Vocabulary

As noted above, key vocabulary terms need to be discussed and defined by students again and again, allowing them to build and refine their understanding of each concept and the connections between them.

- Transformation
- Rigid Motion
- Congruence
- Translation
- Reflection
- Rotation

- Line of Reflection
- Center of Rotation
- Orientation
- Dilation
- Scale Factor
- Similarity
- Proportional
- Center of Dilation

Geometry: Triangles

Overview and Progressions of Learning

Elementary School

Triangles are a focus of geometric thinking as far back as kindergarten, as students learn to identify, compose, and decompose shapes through second grade. In fourth and fifth grades triangles are classified by angles and sides and students use triangles to solve real-world problems about perimeter and area.

Middle School

In the middle grades, students find the area of triangles, explore the possible triangles that could be constructed given specific side lengths and angle measures, and they establish facts about interior and exterior angles of a triangle. They also use similar triangles to describe slope, understand congruence through rigid motion, and use the Pythagorean Theorem to solve problems involving right triangles.

High School

In high school, students build on prior learning as they deepen their understanding of congruence to solve problems, perform transformations, and construct congruence proofs based on generalized criteria. They solve real-world problems involving properties of similar triangles, including special right triangles, and they use the concept of similarity to generalize trigonometric ratios. Students also connect prior geometric knowledge to prove a number of theorems about triangle relationships. Working with triangles not only solidifies student understanding about triangles as geometric objects, but also provides opportunities for students to deepen their understanding of essential geometric ideas like congruence, similarity, and proof.

Content Supports

Listed below are five strategies to build students' conceptual understanding and procedural fluency related to *Triangles*. The five strategies include:

- Real-World Scenarios
- Conceptual and Procedural Balance
- Collaborative Problem-solving
- Vertical Planning
- Daily Formative Assessment

Real World Scenarios

There are a large number of geometry standards that involve triangles, and these standards include priority content for the course. Several important geometric themes are developed more deeply as students work with triangles including congruence, similarity, proof, and trigonometry. In light of these important themes, the use of real-world scenarios and application problems must be thoughtfully considered.

Some geometry standards involving triangles call explicitly for real-world contexts and other standards are easily applied to real-world problems. The topics included in these standards (congruence, similarity, trigonometric ratios, and special right triangles) can be applied to any number of real-world triangular forms including houses, buildings, maps, trees, flagpoles, and the triangular arrangement of any set of objects or people. Given this wide choice of application contexts, it is worthwhile to consider what contexts would most interest students and to allow students to choose contexts when possible.

Real-world scenarios can also be included as students prove triangle congruence or various triangle theorems. These could be contexts about real-world objects and stories, and they can also include triangular objects or models in the classroom or available in the surrounding areas. Non-routine tasks and activities that include real-world objects can also be useful.

- For example, the teacher has a secret triangle ABC in an envelope. Students work in groups, and can ask for information about the triangle including side lengths and angle measures, but they must be specific in their questions. The goal is for students to create a triangle that is congruent (or similar) to triangle ABC with the fewest possible pieces of information.

Activities like this help students understand the different criteria for triangle congruence and similarity, practice geometric construction, develop questioning practice, and elucidate the structure and reasoning of formal proof.

There are also opportunities to design triangle discovery activities that are governed by the limitations of the real world as expressed in geometric theorems.

- For example, students could investigate the Triangle Inequality theorem using given lengths of sticks or straws to build possible triangles. When they can generalize the necessity for the sum of the shorter sides to exceed the length of the longer side, they will have understood the theorem through real-world experience.[MP.7-8]

Discovery activities could also be used to investigate the criteria for triangle congruence, generalize patterns in right triangle trigonometry, or identify ideas about special right triangles.

As students consider trigonometric ratios within right triangles, the use of concrete experiences can help develop student understanding. In addition to the idea that the trigonometric functions are universal properties of angles and determined by angle size, students also need to grasp the different possible pairings of side lengths. Concrete experiences can be arranged where students are asked to occupy different angles of a right triangle and justify trigonometric ratios based on identifying the hypotenuse and adjacent or opposite sides. Activities like this help students identify and name angles, differentiate between sides and angles, and identify appropriate side pairings as the perspective of adjacent and opposite sides change for each acute angle in the triangle.[MP.6,7]

Conceptual and Procedural Balance

In addition to using real-world scenarios to help balance the procedural and conceptual learning outlined in the standards related to triangles, it is important to consider additional ways to attend to this balance. One way to do this is to investigate the different types of expectations in the standards by noting the verbs that are used. Generally, verbs like *apply, use, determine*, and *solve* point to procedural activities that can be assessed through demonstration, while verbs like *explore, explain, justify*, and *prove* indicate more conceptual understandings that can be assessed through explanation and justification. Most of the activities that students engage in will include elements of both types of learning, but if we can be more sensitive to the difference between the two we will be better able to balance and assess them.

The conceptual understanding of what congruence means and how it might be justified is different from the procedures involved in presenting a two column congruence proof. Procedures can often be replicated, but authentic conceptual understanding cannot. Some students may be able to arrange a well-structured proof and lack understanding about what congruence or similarity really means. Other students may understand these ideas conceptually and get lost in the procedures that lead to a structured proof.

Well-engineered classroom discussions that include the whole class and small group talk and an expectation for justification can help strike this balance by developing conceptual understanding for those who need it and justifying procedures for others. The use of thoughtful and open questions can also help by giving teachers the assessment they need to identify when the imbalance between conceptual and procedural learning may be limiting a student's potential.

For some students, there may be a lack of conceptual understanding regarding the structure and limitations of two dimensional space that creates difficulties in their learning. This can include concepts like why two straight lines cannot intersect more than once (unless they are the same line), why the intersection of any number of straight lines describes a circular arc that measures 360 degrees, or what it means that a line of any length contains an infinite number of points and a plane of any area contains infinitely many lines. These important concepts can be assessed through intentional questioning and classroom discussions, and through actively listening to understand student thinking.

For the procedural aspect of the learning, it is important to intentionally allow for and support flexibility as students work. This means giving them choice in the procedures they use, allowing them to make mistakes, and consistently asking them to justify their procedures.[MP.1] Students may be given the opportunity to choose which statements to begin with in a proof, how to justify a geometric relationship, or which trigonometric function to use and which angle to define it for. Allowing for and supporting flexibility can take time and create student struggle, but it builds students' procedural fluency and math practice, and it keeps the focus of instructional time on student thinking which leads to greater retention of learning overall.

Collaborative Problem-solving

The types of problems and investigations that address many triangle standards provide significant opportunities for student collaboration and problem-solving. Presenting groups of students with rigorous problems communicates an expectation for sense-making, peer-learning, discourse, listening, ownership, and responsibility. As the classroom culture grows and the expectation for group work becomes a norm, teachers benefit from the anecdotal assessment they get when observing and listening to group work. Groups can include two to four students and could be asked to prove theorems, create congruence or similarity proofs, or solve problems involving trigonometric ratios. In any event, choices of how to proceed and what representations to create should be left to groups and student-student discourse should be encouraged.

The teacher's role during this time should be to listen, monitor, encourage, and possibly ask questions that address and extend student thinking. If students are struggling or engagement is dwindling, teachers can use catch and release to share student thinking across the whole class and re-engage students.

After collaborative work is complete, students can present and explain their findings and critique or question the work of others.[MP.3] This can include private think time or reflection, gallery walks with or without feedback, as well as whole group discussion.

All discovery or inquiry activities are also opportunities for collaborative problem-solving, such as investigating the Triangle Inequality Theorem or the properties of similarity in right triangles. The tasks could include individual work in addition to collaborating and problem solving with a group, as in generating experimental data individually, and then sharing with others before making generalizations with the group.

Collaborative problem-solving requires certain community norms and takes time, but it also builds essential math practices, keep the focus on student thinking, and increases student ownership of the learning.

Vertical Planning

Geometry content involving triangles spans several domains and multiple clusters within those domains, and these connections should be considered during curricular and lesson planning. Vertical planning can be thought of as "'stacking" standards from different domains or clusters, or teaching more than one standard at a time. Most of the standards related to triangles are in the *Congruence* domain and the *Similarity, Right Triangles, and Trigonometry* domain. These standards populate clusters about transformations, congruence, proving theorems, constructions, similarity, and trigonometry.

The concept of proof expressed in standard GPE.4 are applied and practiced as students construct the proofs and arguments called for in the Congruence standards. Student understanding of the definitions expressed in standard CO.1 are also applied and deepened as students work with standards that explicitly address triangles.

Standard CO.9 includes basic angle theorems and relationships that are applied to the triangle proofs called for in standard CO.10 and SRT.4. The reasoning in standard GPE.5 supports students learning about right triangle trigonometry in SRT.6-8. The constructions and tool use noted in standard CO.12 are extended as students prove triangles congruent in standard CO.7. The treatment of the Pythagorean Theorem in standard GPE.7 prepares students to solve problems involving right triangles called for in SRT.8.

There are many ways to attend to vertical planning, but connections between standards in different domains should be taken into consideration during curricular design and sequencing. In some cases, it may be clear that one standard should be addressed prior to another, but in other cases it may be more a matter of noting the resonance between standards.

Effective vertical planning helps teachers make better use of instructional time overall and can reduce the time between instruction of related content. Regardless of how related domains are sequenced within the curriculum, these connections should be noted and intentionally addressed within planned lessons and activities.

If teachers are intentional about vertical planning, they will gather and respond to assessment with the later standards in mind and will naturally refer back to previously addressed standards when focusing on standards that come later. They will also be more likely to keep language, representations, and procedures consistent and build more effectively on prior learning.

Daily Formative Assessment

Standards involving triangles are connected to priority learning within the Geometry course. Several essential geometric ideas are developed in light of work with triangles, making the use of daily formative assessment critical for student success and effective differentiation. Broad assessment questions like “What does it mean for two triangles to be congruent?”; “Why do all the criteria for triangle congruence have three elements?”; “Why is the sine of all 30 degree angles the same?”; and, “What is trigonometry?” may be essential questions for a student or group of students as they address these standards.

Essential questions should be intentionally planned and asked repeatedly throughout the learning. Students benefit from answering from their own thinking and refining their expression over time, and teachers benefit from the subtle anecdotal assessment they receive. This type of informal formative assessment shapes ongoing instruction. Teachers who include open conceptual questions as part of daily lesson plans develop more effective questioning practice overall.

Daily formative assessment may be formal or informal. It is often anecdotal and generated from classroom discussions with individual students or groups of students. Formative assessment is based on asking good questions, and can be gained through checks for understanding where students respond in the moment either verbally or physically (raising hands, or giving a thumbs up or thumbs down). It can also be generated through exit tickets or reflection questions given at the end of class or at the completion of an activity. In all cases, formative assessment is conducted while students are still learning the material. The assessment should generate data about *how* students are learning the material and what misconceptions are arising, so that teachers have time to adjust instruction and students have additional opportunities to build understanding.

Tasks and activities designed for formative assessment are also an essential part of daily formative assessment.

The following links show some examples of formative assessment tasks and lessons from the Mathematics Assessment Project that involve triangles.

- Circles in Triangles
  https://www.map.mathshell.org/tasks.php?unit=HA08&collection=9&redir=1
- Hopewell Geometry
  https://www.map.mathshell.org/tasks.php?unit=HA05&collection=9&redir=1
- Evaluating Conditions for Congruency
  https://www.map.mathshell.org/lessons.php?unit=9315&collection=8&redir=1
- Deducting Relationships: Floodlight Shadows
  https://www.map.mathshell.org/lessons.php?unit=9305&collection=8&redir=1

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not mastering the standard. What is it that they do not understand? And how far back do their misconceptions go in the prerequisite standards? If we can identify the missing concepts, we can begin to understand why they are struggling with the current standard. Students may also begin to grasp some of the conceptual parts of the standards and yet struggle with new vocabulary, deeper connections between ideas, or additional notation.

For students who struggle with standards related to triangles, it is important to be very clear about the main concepts being addressed and to assess student understanding of prior related content. These standards require that students integrate their understanding of significant geometric ideas with formal definitions, new procedures, and new structures such as formal proof. Within the focus of each standard, assessment questions and the expectation for student justification can determine the shape of instruction and any necessary differentiation or remediation. Summative assessment can help measure student learning after the learning is finished, but formative assessment informs us about student learning while the learning is still going on and while there is still time to address difficulties in later instruction.

In addition to the broad formative assessment questions noted above, the following questions can be helpful in assessing misconceptions and developing understanding and procedural fluency related to triangles.

- What is a triangle and what are its parts?
- What are some things that are true about all triangles?
- How do we classify triangles? What are some different types of triangles?
- When can we apply the Pythagorean Theorem and what does it help us find?
- What do we use theorems for in geometry?
- What is congruence?
- How can we define congruence in terms of transformations?
- What are the different ways we can determine or justify triangle congruence?
- Why don't we need to know every measure in a triangle to determine if it is congruent to another triangle?
- Why are two congruent corresponding sides and the non-included angle not sufficient criteria for triangle congruence?
- What are congruent corresponding angles not sufficient criteria for triangle congruence?
- What does it mean for two triangles to be congruent?
- What does it mean for two triangles to be similar?
- How do we show that two triangles are similar?
- What can we conclude about the perimeter and area of congruent triangles?
- What can we conclude about the perimeter and area of similar triangles?
- What does the Triangle Inequality Theorem tell us?
- How do you determine if three side lengths can make a triangle?
- When do three sides lengths not make a triangle?
- What is an altitude and what are its properties?
- What is created when we draw an altitude to the hypotenuse of a right triangle? Can you make and label a diagram to explain?
- What do trigonometric functions like sine, cosine, and tangent depend on? What do they describe?
- What do you know about the side lengths of special right triangles (30-60-90 and 45-45-90), and how could you prove it?

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including the following:

- Revoicing – "So you're saying that…."MP.6
- Repeating – "Can you repeat Hayley's reasoning?"MP.2
- Reasoning – "Do you agree or disagree? Why?"MP.3
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talkMP.1

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work. Academic vocabulary actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.MP.6

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.MP.3 The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are "Why?"; "What does that mean" ; and "How do you know?" These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

With the large amount of content related to triangles, language is particularly important as students try to integrate all the vocabulary terms and their relationships and teachers seek to assess student thinking. Group activities that encourage student-student talk are essential and whole class discussions should be intentionally designed and facilitated. It is important that all work with formal proofs, whether proving theorems or triangle congruence and similarity, is accompanied by significant verbal justification and discussion.

Language and discussion about trigonometry can be a source of confusion and should be closely monitored. For this and later courses, students need to understand that sine, cosine, and tangent are properties of angles, and the language should reflect that, as in the sine of angle A or the sine of a 30 degree angle. The angles retain the trigonometric values whether they are in a triangle or not, and the trigonometric values are a function of the angle measure and universal for that angle measure. They should also begin to understand that the sine, cosine, or tangent of an angle is equivalent to a numerical value that may be found using a side ratio from a right triangle or from other means. In the notation, students should know that *sin* or *cos* by itself has no meaning, but must always be included with an angle measure whether defined or undefined.

Vocabulary

As noted above, key vocabulary terms need to be discussed and defined by students again and again, allowing them to build and refine their understanding of each concept and the connections between them.

- Triangle
- Angle
- Side
- Vertex
- Hypotenuse
- Leg
- Theorem
- Triangle Angle Sum Theorem
- Isosceles Triangle Theorem (and converse)
- Midsegment Theorem
- Triangle Proportionality Theorem
- Angle Bisector Theorem
- Pythagorean Theorem
- Triangle Inequality Theorem
- Hinge Theorem
- Interior Angle
- Exterior Angle
- Congruent
- Rigid Motion
- Criteria
- Corresponding Parts
- SAS Postulate (Side Angle Side)
- SSS Postulate (Side Side Side)
- ASA Postulate (Angle Side Angle)
- AAS Postulate (Angle Angle Side)
- HL (hypotenuse leg) Postulate
- CPCTC (Corresponding parts of congruent triangles are congruent)
- Similar

- Dilation
- AA (Angle Angle) Similarity
- Proportional
- Perimeter
- Area
- Altitude
- Median
- Right Triangle
- Geometric Mean
- Trigonometric Ratio
- Sine
- Cosine
- Tangent
- Adjacent Side
- Opposite Side
- Hypotenuse
- Acute Angle
- Special Right Triangle
- 30-60-90 Triangle
- 45-45-90 Triangle

Geometry: Quadrilaterals and Other Polygons

Overview and Progressions of Learning

Elementary School

Students compare and compose basic shapes in kindergarten, and they identify basic quadrilaterals by name in first and second grades and decompose shapes to informally consider area in second grade. In third grade, shapes are classified by their attributes and place within an informal hierarchy based on those attributes. Students in fourth grade begin to classify shapes by parallel and perpendicular lines and lines of symmetry; and, in fifth grade they classify polygons by their angles and sides and place them in a hierarchy based on their properties.

Middle School

Middle school students investigate the interior angles of polygons, solve problems about shapes graphed on a coordinate plane, and examine area by composing and decomposing polygons. They learn about similarity through proportional reasoning and scale drawing and identify basic angle pair relationships. In eighth grade, students understand congruence through rigid motion transformations of shapes and figures.

High School

In high school Geometry, students extend prior learning about polygons to prove theorems about parallelograms and prove the nature of quadrilaterals by their properties. They develop formulas to generalize their understanding of interior and exterior angles in polygons, investigate different types of symmetry, and solve real-world problems about perimeter and area. The study of quadrilaterals and other polygons is an essential geometry topic, as it provides numerous opportunities for students to deepen key understandings such as congruence, similarity, angle relationships, area, and geometric proof.

Content Supports

Listed below are five strategies to build students' conceptual understanding and procedural fluency related to *Quadrilaterals and Other Polygons*. The five strategies include:

- Real-World Scenarios
- Conceptual and Procedural Balance
- Collaborative Problem-solving
- Vertical Planning
- Daily Formative Assessment

Real-World Scenarios

Working with quadrilaterals and other polygons provides additional opportunities for students to develop several important geometric understandings such as properties of polygons, proof, and congruence. The use of real-world scenarios continues to be a high leverage practice that deepens students' conceptual understanding of the content and connects the geometry to the many quadrilaterals and other polygons that populate the world around us.[MP.4]

Standard GPE.7 asks students to solve problems about areas and perimeters of polygons on the coordinate plane. There are numerous possible applications for this standard including buildings, floor plans, ball fields, computer screens, building blocks, design patterns, and packing boxes. Stories about people moving within any of these contexts can create more complicated figures and problems that could relate to rates of speed or optimal pathways. All standards that involve work with quadrilateral can be applied to real-world contexts. This could include problem-solving about areas about regular polygons, interior and exterior angles, or different kinds of symmetry. The process of modeling the real world with mathematics demands flexibility of reasoning and representation that is essential to a well-rounded math practice.[MP.2,4]

Real-world applications can be included as students address proofs about parallelograms in standards CO.11. Problems including proofs can be presented without a context, but the use of context can increase engagement and activate student reasoning more deeply. Students benefit when given the opportunity to create graphical representations from verbal descriptions as opposed to being given diagrams that are already complete. The process of creating a detailed and scaled representation on the coordinate plane engages their geometric and spatial reasoning. Real-world contexts can also be generated by hands on models, various objects, or architectural features from within the classroom or school grounds.

Conceptual and Procedural Balance

It is likely that in Geometry students have had significant opportunities to develop their understanding about important geometric ideas like congruence, similarity, and properties of plane figures. In order for students to continue to build these understandings, it is important to be attentive to the balance between conceptual and procedural learning in daily instruction. In addition to using real-world scenarios to help strike this balance, it may be useful to investigate the different types of expectations in the standards by noting the verbs that are used. Generally, verbs like *apply, use, find*, and *compute* point to procedural activities that can be assessed through demonstration, while verbs like *prove* and *develop* indicate more conceptual understandings that can be assessed through explanation and justification. Most of the activities students engage in will include elements of both types of learning, but if we can be more sensitive to the difference between the two, we will be better able to balance and assess them.

The conceptual understanding of what it means to prove a theorem or to prove the nature of a quadrilateral based on reasoning about its properties is different from the procedures involved in presenting a two column proof. Procedures can often be replicated, but authentic conceptual understanding cannot. Some students may be able to arrange a well-structured proof and lack understanding about what the reasoning implies. Other students may understand these ideas conceptually and get lost in the procedures that lead to a structured proof. Well-engineered classroom discussions that include the whole class and small group talk and an expectation for justification can help support this balance by developing conceptual understanding for those who need it and justifying procedures for others. Having students express their reasoning verbally and informally can be an important step in moving them toward meaningful construction of formal proof. The use of thoughtful and open questions can also help by giving teachers the assessment they need to identify when the imbalance between conceptual and procedural learning may be limiting a student's potential.

For some students, there may be a lack of conceptual understanding regarding the structure and limitations of two dimensional space that creates difficulties in their learning. This can include concepts like why two straight lines cannot intersect more than once (unless they are the same line), why the intersection of any number of straight lines describes a circular arc that measures 360 degrees, or what it means that a line of any length contains an infinite number of points and a plane of any area contains infinitely many lines. These important concepts can be assessed through intentional questioning and classroom discussions, and through actively listening to understand student thinking.

For the procedural aspect of the learning, it is important to intentionally allow for and support flexibility as students work. This means giving them choice in the procedures they use, allowing them to make mistakes, and consistently asking them to justify their procedures.[MP.1] Students may be given the opportunity to choose which statements to begin with in a proof, how to justify a geometric relationship, or how to decompose a polygon to find its area. Regarding the formulas to find interior angle measure or the area for different polygons, it is important to remember that the use of a formula without understanding is not advised.[MP.6] Students should be encouraged to think about patterns in angles and area and consider different ways to find their measures in addition to applying a formula. When using formulas, students should be asked to justify the generalization represented in the formula. Allowing for and supporting flexibility can take time and create student struggle, but it builds students' procedural fluency and math practice, and it keeps the focus of instructional time on student thinking which leads to greater retention of learning overall.

Collaborative Problem-solving

Problems and investigations that include quadrilaterals and polygons provide significant opportunities for student collaboration and problem-solving. Presenting groups of students with rigorous problems communicates an expectation for sense-making, peer-learning, discourse, listening, ownership, and responsibility.

As the classroom culture grows and the expectation for group work becomes a norm, teachers benefit from the anecdotal assessment they get when observing and listening to group work. Groups can include two to four students, and they could be asked to prove theorems about parallelograms, create proofs that classify quadrilaterals, or solve real-world problems about perimeter and area. In any event, choices of how to proceed and what representations to create should be left to groups, and student-student discourse should be encouraged.

The teacher's role during this time should be to listen, encourage, and possibly ask questions that address and extend student thinking. If students are struggling or engagement is dwindling, teachers can use catch and release to share student thinking across the whole class and re-engage students.[MP.1]

After collaborative work is complete, students can present and explain their findings and critique or question the work of others.[MP.3] This can include private think time or reflection, gallery walks with or without feedback, as well as whole group discussion.

All discovery or inquiry activities are also opportunities for collaborative problem solving. This could be applied to investigations about interior and exterior angles of polygons or areas of regular polygons. The tasks could include individual work in addition to collaborating and problem solving with a group, as in generating experimental data individually and then sharing with others.

Collaborative problem-solving requires certain community norms and takes time, but it also builds essential math practices and is another powerful way to keep the focus on student sense-making and increase student ownership of the learning.

Vertical Planning

Quadrilaterals and other polygons connect to learning in almost all geometry domains, including congruence, similarity, expressing geometric properties with equations, measurement, and modeling. Vertical planning can be thought of as "stacking" standards from different domains and clusters, teaching more than one standard at a time, or being thoughtful about connects between current standards and past or futures standards.

The concept of proof expressed in standard GPE.4 is applied and practiced as students construct the proofs and arguments called for in standard CO.11. Student understanding of the definitions expressed in standard CO.1 are also applied and deepened during problem solving with quadrilaterals and other polygons.

The theorems and relationships indicated in standards CO.9 and GPE.5 are essential to understanding the properties of quadrilaterals as well as proofs that classify the various quadrilaterals. Student learning about using the Pythagorean Theorem to calculate distance on the coordinate plane from standard GPE.7 is also crucial to proof statements about congruent sides of quadrilaterals. Concepts about interior angles of triangles that are developed in CO.10 are extended as students investigate and generalize patterns about interior and exterior angles of polygons in general, and learning about the area of triangles supports understanding about areas of quadrilaterals.

Finally, quadrilaterals and their properties provide a foundation for learning about and solving problems with three-dimensional objects. This includes work with various prisms as well as two-dimensional cross sections of three-dimensional objects.

There are many ways to attend to vertical planning, but connections between standards in different domains and clusters should be taken into consideration during curricular design and sequencing. In some cases, it may be clear that one standard should be addressed prior to another, but in other cases it may be more a matter of noting the resonance between standards or using one standard to solidify learning about another.

Effective vertical planning helps teachers make better use of instructional time overall and can reduce the time between instruction of related content. Regardless of how related domains are sequenced within the curriculum, these connections should be noted and intentionally addressed within planned lessons and activities.

If teachers are intentional about vertical planning, they will gather and respond to assessment with the later standards in mind and will naturally refer back to previously addressed standards when focusing on standards that come later. They will also be more likely to keep language, representations, and procedures consistent and build more effectively on prior learning.

Daily Formative Assessment

Students continue to develop key geometric concepts as they work with quadrilaterals and other polygons, making the use of daily formative assessment critical for student success and effective differentiation. Broad assessment questions, like "What is a parallelogram?"; "What are some key properties that help us classify quadrilaterals?"; and "What is the difference between perimeter and area?" may be essential questions for a student or group of students as they address these standards.

Essential questions should be intentionally planned and asked repeatedly throughout the learning. Students benefit from answering from their own thinking and refining their expression over time, and teachers benefit from the subtle anecdotal assessment they receive. This type of informal formative assessment shapes ongoing instruction. Teachers who include open conceptual questions as part of daily lesson plans develop more effective questioning practice overall.

Daily formative assessment may be formal or informal. It is often anecdotal and generated from classroom discussions with individual students or groups of students. Formative assessment is based on asking good questions, and can be gained through checks for understanding where students respond in the moment either verbally or physically (raising hands, or giving a thumbs up or thumbs down). It can also be generated through exit tickets or reflection questions given at the end of class or at the completion of an activity.

In all cases, formative assessment is conducted while students are still learning the material. The assessment should generate data about *how* students are learning the material and what misconceptions are arising, so that teachers have time to adjust instruction and students have additional opportunities to build understanding.

Tasks and activities designed for formative assessment are also an essential part of daily formative assessment.

The following links show some examples of formative assessment tasks and lessons from the Mathematics Assessment Project that involve quadrilaterals.

- Floor Pattern
  https://www.map.mathshell.org/tasks.php?unit=HA10&collection=9&redir=1
- Square
  https://www.map.mathshell.org/tasks.php?unit=HA22&collection=9&redir=1
- Evaluating Statements About Length and Area
  https://www.map.mathshell.org/lessons.php?unit=9310&collection=8&redir=1

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not mastering the standard. What is it that they do not understand? And how far back do their misconceptions go in the prerequisite standards?

If we can identify the missing concepts, we can begin to understand why they are struggling with the current standard. Students may also begin to grasp some of the conceptual parts of the standards and yet struggle with new vocabulary, the practice of proof, or additional notation.

For students who struggle with learning about quadrilaterals, it is important to attend to the key geometric concepts and identify what prior learning is not culminating in the current learning. Many concepts related to quadrilaterals and other polygons are extensions of learning about triangles. Key concepts include the idea and practice of formal proof, the concept of parallelism, and the idea that we classify shapes by properties which may depend on or imply other properties within a hierarchy.

Some students may require additional hands-on experience or modeling with manipulatives in order to effectively transfer and apply prior knowledge to the topic of polygons in general. Within the focus of each standard, assessment questions and the expectation for student justification can determine the shape of instruction and any necessary differentiation or remediation. Summative assessment can help measure student learning after the learning is finished, but formative assessment informs us about student learning while the learning is still going on and while there is still time to address difficulties in later instruction.

In addition to the broad formative assessment questions noted above, the following questions can be helpful in assessing misconceptions and developing understanding and procedural fluency related to quadrilaterals and other polygons.

- What is a parallelogram?
- Are all rectangles parallelograms? Why or why not?
- Why are opposite angles in a parallelogram congruent?
- Why are opposite sides of a parallelogram congruent?
- What other quadrilaterals are also parallelograms? Which are not?
- What is the main defining property of a rectangle? A square? A rhombus? A trapezoid?
- When is a square also a rhombus?
- How would you determine the measure of an interior angle of an n-gon?
- How would you determine the measure of an exterior angle of an n-gon?
- Why does the formula for interior angles of a polygon work? What does each part of the formula represent?
- What are some examples of real life objects that have symmetry?
- What is the difference between line and point symmetry? Can a figure have both? Can you give an example?
- How many lines of symmetry does a rectangle have? How do you know you have found them all?
- What is perimeter?
- What is area?
- How are area and perimeter different?
- How do you find the area of quadrilaterals? A regular polygon?
- How would you find the area of a trapezoid if you forgot the formula?
- How would you find the area of a regular polygon if you forgot the formula?
- Why does the formula for the area of a polygon work?

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including the following:

- Revoicing – "So you're saying that…."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own

- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important. For most students, it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work. Academic vocabulary actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

Encourage Student Sense-making

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are "Why?"; "What does that mean?"; and "How do you know?" These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

The expectation for justification is of particular importance as students learn to differentiate different quadrilaterals by necessary properties as well as visually. When students use appropriate names of the different shapes, it should be a consistent practice to have them justify how they know they are naming the shape correctly and ensure that definitions are tied to properties.

It is common for students to struggle to understand and express the hierarchy in properties when classifying or defining quadrilaterals. They may mistakenly say that a shape is a rectangle because opposite sides are congruent or that a square is not a rhombus because it has four right angles. These misconceptions need to be brought out, challenged, and discussed.[MP.3,6] Understanding is never a matter of memorization, although it can help, it is the product of thinking, reasoning, justifying, and engaging the right to change one's mind.

Regarding any formulas related to quadrilaterals, it is important that their use is connected to planned discussions about their meaning and why they work. Formulas are generalizations of mathematical principles, and users are more empowered when they understand the meanings within those principles and why the formulas work. Students can be asked to explain what relationships the formulas represent, what each element of the formula means, and also how they would find the desired measures without the formulas.

Vocabulary

As noted above, key vocabulary terms need to be discussed and defined by students again and again, allowing them to build and refine their understanding of each concept and the connections between them.

- Polygon
- Regular Polygon
- Quadrilateral
- Parallelogram
- Rectangle
- Square
- Rhombus
- Trapezoid
- Kite
- Adjacent Angles
- Opposite Angles
- Adjacent Sides
- Opposite Sides
- Diagonal
- Bisect
- Vertex
- Interior Angle
- Exterior Angle
- Symmetry
- Point Symmetry
- Line Symmetry
- Rotational Symmetry
- Perimeter
- Area
- Distance
- Pythagorean Theorem
- Apothem

Geometry: Circles

Overview and Progressions of Learning

Elementary School

Students interact with circles as early as kindergarten. In first grade, they partition circles and compose and decompose shapes; and, in second and third grades, they consider the importance of equal parts and use equal parts of a whole to represent fractions. In fourth grade, students understand angles as the measurement of part of a circle in degrees and consider an informal definition of a circle.

Middle School

In middle school, students identify the radius and diameter of a circle, learn to calculate the area and circumference of a circle, and apply all their knowledge of circles to solve real world problems

High School

In high school Geometry, students consider circles in light of big ideas like congruence and similarity, understand and apply theorems about circles, find arc lengths and areas of sectors or circles, and use circle relationships to model real world problems. Definitions for circles and related geometric figures become more precise, and students consider the connections between terms more deeply as they formalize understandings from prior courses. In post-Geometry courses, students represent circles with equations as they study conic sections and use the unit circle to understand trigonometric functions.

Content Supports

Listed below are five strategies to build students' conceptual understanding and procedural fluency related to *Circles*. The five strategies include:

- Real-World Scenarios
- Key Concepts
- Vertical Planning
- Multiple Representations and Tools
- Daily Formative Assessment

Real-World Scenarios

As students apply and extend previous understandings of circles and angles from elementary and middle school, solving problems that include real-world scenarios provides important opportunities to deepen their conceptual understanding and communicate that understanding to teachers and peers. Circular forms abound in the world around us, and relationships between measures within those forms can generate engaging problem-solving experiences.[MP.4]

Standard C.5 extends the area and circumference work in middle school to arc and length and areas of sectors, both of which apply easily to real-world application, while C.2 involves relationships within and between circles that can be used to model real world situations. The final standards in the domain, C.3-4, focus on constructing circles and tangent lines under given conditions and constructing circles within and around certain shapes. These constructions relate directly to the real-world because they play out in real space through concrete experience, but they can also be applied to a variety of problem-solving contexts.

Problems involving radius, diameter, arc length, circumference and area can be applied to any number of real world scenarios including car or bicycle tires, carousels, the area lit from an overhead lamp, the area occupied by a dog tied to a post, and pizza pies. Applications that touch on angles and figures within a circle can help model the movement of a player in a soccer or basketball game, or the optimal position to score in soccer from the sideline based on angular access to the goal.

As students consider the given context, they may have to interpret and label a given diagram or create their own diagram to help them reason about the problem.[MP.2, 4] They may also have to engage prior learning about angle relationships, properties of polygons, or the Pythagorean theorem. While such problems can be given with a simple diagram and without a context, students benefit from reasoning within a context as it activates conceptual thinking and real world experience and develops their mathematical practice in general. Rich problems should include non-routine presentations, multiple entry points, and multiple solution paths.[MP.1]

Consider this problem from Phillips Exeter Academy's Math 2 problem set:

> *A circular table is placed in the corner of a room so that it touches both walls. A mark is made on the edge of the table, exactly 18 inches from one wall and 25 inches from the other. What is the radius of the table?*

Or this problem from the Mathematics Assessment Project:

- Temple Geometry

Other applications to consider for the standards in this domain include problems that can be investigated in real space using hands on items.[MP.5] Investigating circular objects or objects with circular properties such as cylinders and using them to solve problems can be helpful. Circles, and all the related measures and angles, can also be formed by students standing or moving and using a length of rope fixed to a center point. While precise measurements can be difficult in these circumstances, concrete experience can be very useful to help students understand the meaning of vocabulary words, key concepts, and specific relationships.

Key Concepts

As students work to grasp the content addressed by the standards in this domain, one strategy that can help to ensure a balance between procedural and conceptual learning is to plan instruction around the key concepts that dominate the mathematics. A solid grasp of these ideas will help students navigate this content with confidence. Confusion about key concepts will produce misconceptions and uncertainty. Key concepts also help to clarify learning goals, generate formative assessment, and optimize instructional time. Key concepts in the *Circles* domain include:

- Congruence and Similarity
- Dimensionality
- Parallelism
- Perpendicularity
- Infinity

Two important ideas that are present throughout the Geometry course are congruence and similarity, and they are also key concepts that contribute to a full understanding of circles. An understanding (and justification) that all circles are similar gives meaning to the proportionality addressed in C.5, and the universality of the relationships indicated by C.2. Criteria for congruence of circles requires an understanding that circles are defined by the measure of their radii and that all radii within a circle (or in congruent circles) are congruent. As students address the standards in this domain, attending to and questioning about congruence and similarity can help connect specific learning to larger concepts and can give teachers essential assessment about ongoing misconceptions.

Regarding dimensionality, students need to understand that while circles are two dimensional, they also include one dimensional (linear) measurement that can be straight (as in radii) or curved (as in circumference or arcs). Students need to differentiate between these one dimensional measures within a circle and the area of circles or segments which is a two dimensional measurement.

Furthermore, the circles investigated in application tasks may be parts of three dimensional objects such as cylinders or spheres. As students learn to navigate these different dimensions, questions about the dimensionality of specific objects or relations between dimensions can help students make deeper connections, clarify their thinking, and inform teachers about gaps in understanding.

Key concepts like infinity, dimensionality, parallelism, and perpendicularity can also increase or inhibit access to the learning about circles. The concepts that every circle is a locus of infinitely many points and includes infinitely many radii extending from a center to those points are essential for meaningful work with circles and related problem solving. A full understanding of tangent lines and problem solving using right triangles within a circle rests on a clear understanding of the meaning of parallel and perpendicular.

It is not uncommon for high school Geometry students to have an incomplete understanding of these key concepts, and ongoing questioning and discussion about them and the role they play in the standards helps deepen student understanding and gives necessary formative assessment that teachers can use to adjust instruction.

Vertical Planning

The teaching and learning in every domain in the Geometry course needs to be planned and implemented with attention to connections between standards across domains. Vertical planning can be thought of as "stacking" standards from different domains or teaching more than one standard at a time. The standards in the *Circles* domain connect to several other domains as they develop broad strands of learning including congruence, proof, and geometric problem solving.

In the *Congruence* (CO) domain, standards CO.1, 4 and 13 contribute to a foundational understanding of circular figures and support the reasoning required for all circles standards. The theorems addressed in standard CO.9 form a basis for reasoning about the angle relationships noted in C.2 and also develop the broader theme of congruence.

Standard CO.12 includes the tool use and concepts around construction that will be necessary to address standard C.3-4, and standard GPE.7 helps students deepen their understanding of the Pythagorean Theorem which has a role in the problem-solving with circles implied by standards C.2-3. Standards GPE.1 and GPE.4 involve the equation of a circle and problem-solving with circles on the coordinate plane. Finally, the topics of congruence and similarity as they apply to circles and related figures are also significantly developed in the *Similarity, Right Triangles, and Trigonometry* domain, particularly standards SRT.2, SRT5, and SRT8.

There are many ways to attend to vertical planning, but connections between standards in different domains should be taken into consideration during curricular design and sequencing. In some cases, it may be clear that one standard should be addressed prior to another, but in other cases it may be more a matter of noting the resonance between standards.

Effective vertical planning helps teachers make better use of instructional time overall and can reduce the time between instruction of related content. Regardless of how related domains are sequenced within the curriculum, these connections should be noted and intentionally addressed within planned lessons and activities. If teachers are intentional about vertical planning, they will gather and respond to assessment with the later standards in mind and will naturally refer back to previously addressed standards when focusing on standards that come later. They will also be more likely to keep language, representations, and procedures consistent and build more effectively on prior learning.

Multiple Representations and Tools

As in most geometric learning, the use of multiple representations and visual learning is essential for students as they address the standards in the *Circles* domain.

Individual white boards are a powerful tool for student experimentation and teacher assessment.[MP.5] They are low investment for students and easily erased. They also provide teachers with a semi-permanent record of student thinking that can be communicated, justified, or questioned. White boards can be used to assess student understanding about a geometric object or relationship by simply asking a student to draw the figure or relationship and explain.

For example, some students mistakenly think that all intersecting lines are perpendicular (whereas parallel lines “don’t touch”). Asking a student to draw a pair of perpendicular lines and justify can quickly surface this misconception. Practicing drawing circles and labeling important features like radii, arcs, and angles can support every standard in the domain and helps keep the focus of learning on student thinking. It is important to recognize the difference between the event of solving a problem with a given diagram, and having a student create a legitimate diagram and use it to solve a problem.

Regarding constructions, the simple tools of compass and straightedge can often be problematic for students.[MP.5] They need significant practice, patient instruction, and opportunities to help and teach others how to use the tools. Students also benefit from being asked to consider what it means that all constructions can be created with only these two tools.[MP.8]

The power of a circle or arc as a measurement of distance in multiple directions is an essential geometric idea that gives meaning to the practice of constructions and to all the *Circles* standards.

- Virtual geometry software such as *Geogebra* can help students understand constructions more quickly and allow constructions to become an investigation and justification tool for students rather than just a one-time creation of different objects.

Finally, the use of concrete experiences can help students understand circles and the relationships within circles.[MP.4-5] This can be done by marking the floor with tape or having students represent parts of a circle, ends of a line segment, or vertices of a triangle. A rope of fixed length can be used as a radius to create a “human compass.” As with all representations, not all students may need concrete experiences to understand the geometry, but experiencing geometric relationships in real space and seeing the figures from within while also defining specific features builds conceptual and spatial understanding that some students may not acquire through other presentations.

Daily Formative Assessment

The standards in the *Circles* domain extend learning about circles from previous grades and also connect to several important geometry topics. Broad assessment questions, like "What is a circle?"; "What are some important parts of a circle?"; and, "What are some important relationships within a circle?" may be essential questions for a student or group of students as they address these standards.

Essential questions should be intentionally planned and asked repeatedly throughout the learning. Students benefit from answering from their own thinking and refining their expression over time, and teachers benefit from the subtle anecdotal assessment they receive. This type of informal formative assessment shapes ongoing instruction. Teachers who include open conceptual questions as part of daily lesson plans develop more effective questioning practice overall.

Daily formative assessment may be formal or informal. It is often anecdotal and generated from classroom discussions with individual students or groups of students. Formative assessment is based on asking good questions, and can be gained through checks for understanding where students respond in the moment either verbally or physically (raising hands, or giving a thumbs up or thumbs down). It can also be generated through exit tickets or reflection questions given at the end of class or at the completion of an activity. In all cases, formative assessment is conducted while students are still learning the material. The assessment should generate data about *how* students are learning the material and what misconceptions are arising, so that teachers have time to adjust instruction and students have additional opportunities to build understanding.

Tasks and activities designed for formative assessment are also an essential part of daily formative assessment.

The following links show some examples of formative assessment tasks and lessons from the Mathematics Assessment Project that involve circles.

- Circles and squares
- Inscribing and Circumscribing Right Triangles
- Solving Problems with Circles and Triangles
- Circles in Triangles

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why students are not mastering the standard. What is it that they do not understand? And how far back do their misconceptions go in the prerequisite standards? If we can identify the missing concepts, we can begin to understand why they are struggling with the current standard.

For students who struggle with circles, it is important to identify how they think about the key concepts and prior fluencies implied by the standards. This is primarily done through productive discourse and intentional task selection. Misconceptions about key concepts or the use of unreliable procedures can create difficulties for students if left undiscovered, but if brought to the surface they can lead to deeper connections and allow students to build their mathematical practice.

In addition to the broad formative assessment questions noted above, the following questions can be helpful in assessing misconceptions and developing understanding and procedural fluency related to circles.

- What is the definition of a circle?
- What is a radius? A diameter?
- What is a chord?
- What is a secant?
- What is a tangent?
- What is the difference between a chord and a secant? A chord and a tangent?
- What are some things you know about two congruent circles?
- How many radii are there in a circle?
- What is the relationship between the radius of a circle and the arc intercepted by a central angle?
- Circle A has radius AB and angle CAB intercepts arc CB. If radius AB doubles in length, what happens to arc CB and angle CAB?
- What is a sector of a circle? What are some parts of a sector?
- How do you find the area of a sector?
- Circle O has central angle AOC and Inscribed angle ABC. Why is the measure of angle ABC half the measure of angle AOC?
- What units do we use to measure arcs in a circle? Why?
- What units do we use to measure central and inscribed angles in a circle?
- Why is the inscribed angle whose endpoints lie on the diameter equal to 90 degrees?
- What measure does the circumference and area of a circle depend on?
- What is Pi? What does Pi represent?
- Why does a tangent line form a right angle with the radius drawn to the point of tangency?
- When making constructions, which tool is used to measure?

- What geometric constructions do you know how to make?

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including the following:

- Revoicing – "So you're saying that…."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work. Academic vocabulary actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

Encourage Student Sense-making

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are "Why?"; "What does that mean?"; and "How do you know?" These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

Some language to be intentional about during the learning in this domain is the distinction between '"n the circle, "on the circle," and "outside the circle." Not only does this language support a full understanding of the structure and definition of a circle, but it also comes into play during modeling and problem solving. Students that struggle with receiving or using this language may have underlying misconceptions about circles that need to be addressed.

Vocabulary

As noted above, key vocabulary terms need to be discussed and defined by students again and again, allowing them to build and refine their understanding of each concept and the connections between them.

- Circle
- Center
- Radius (Radii)
- Diameter
- Arc
- Chord
- Secant
- Tangent
- Congruent
- Concentric
- Similar
- Intercepted Arc
- Sector
- Proportional
- Central Angle
- Inscribed
- Circumscribed
- Parallel
- Perpendicular
- Circumference
- Area
- Pi
- Infinite

Geometry: Three-Dimensional Figures

Overview and Progressions of Learning

Elementary School

Students begin differentiating between two and three-dimensional objects in kindergarten. In first grade, they compose solids such as prisms, cones, and cylinders. In second grade they classify three-dimensional shapes by faces and vertices. By third grade, students begin to solve problems involving volume.

Middle School

In middle school, students find the volume of the right rectangular prisms and cylinders. They then go on to solve real-world problems involving the volumes of cylinders, cones, and spheres. They also create and use nets to understand the connections and differences between two- and three-dimensional space and define attributes of right rectangular prisms, cylinders, cones, spheres, and pyramids.

High School

High school Geometry students build on prior learning as they continue to use nets to connect two and three-dimensional objects. As students mature, they are increasingly able to conceptualize relationships between two and three-dimensional space and imagine solids of various orientations. Students continue to solve real-world problems about volume and surface area as well as design problems within a context. They also consider the symmetry, congruence, and similarity of solids as they engage in application tasks that include three-dimensional objects.

Content Supports

Listed below are five strategies to build students' conceptual understanding and procedural fluency related to *Three-Dimensional Solids*. The five strategies include:

- Real-World Scenarios
- Hands-On Learning
- Space and Dimensions
- Justifying Procedures With Concepts
- Daily Formative Assessment

Real-World Scenarios

The standards related to three-dimensional figures extend much of the prior learning of two-dimensional geometry into the space of three-dimensions, making it particularly applicable to real-world problems.[MP.4] Three-dimensional space is what students move through every day as they handle three-dimensional objects and solve three-dimensional problems. The use of real-world scenarios helps activate students' reasoning and engages thinking about authentic situations that could be modeled with geometry.

Standards GMD.1-4 connect directly to real-world applications involving volume and surface area of solids. There are many possible contexts for this mathematics, but the most effective contexts are relevant or interesting to students, are puzzling or create the impetus for inquiry, and/or include multiple phases of matter including gas and liquid. Contexts can be relevant because they include ideas that are meaningful to students, like sports, candy, or animals, or because they relate to scenarios that students have had personal experience with, like fish tanks or gift boxes. Contexts that create inquiry often include a question or prediction about what is going to happen under given conditions, like how many times one volume will fit into another or how long it will take to fill a tank with water. Phases of matter are interesting because most liquids and gases take the shape of their container and allow for displacement, making for non-routine volume calculations.

Consider this problem from Phillips Exeter Academy's Math 1 problem set:

> *The base of a rectangular tank is three feet by two feet, and the tank is three feet tall. The water in the tank is currently nine inches deep.*
>
> - *How much water is in the tank?*
> - *The water level will rise when a one-foot metal cube (denser than water) is placed on the bottom of the tank. By how much?*
> - *The water level will rise some more when a second one-foot metal cube is placed on the bottom of the tank, next to the first one. By how much*

Or this problem from the Mathematics Assessment Project:

> *Propane Tanks*

Standard MG.1-3 call for the use of design problems and modeling situations. These types of problems are inherently real-world and often include multiple interrelated relationships or constraints, such as cost or availability of materials, sales price, volume constraints, or time.[MP.4] The optimization required in such problems offer high cognitive demand are a good opportunity to engage students in small group or partner collaboration and discussion.[MP.2]

Hands-On Learning

Another way to activate this content with students is by using hands-on activities where students can interact with and manipulate two- and three-dimensional objects in real space. While students should be expected to use spatial reasoning to think about nets and cross sections abstractly, they can also be asked to design a net that will fold into a particular solid and then actually build that solid by folding the net.[MP.2] Once the solid is created, students can solve problems about measures like surface area or volume using the manipulative as reference.

Hands on models of the various solids are very useful in helping students define the different solids and gain a deeper understanding of their properties, characteristics, similarities, and differences. Problems about symmetries and congruent or similar solids can also be applied to hands on models. Students may find it useful to identify real-world objects within the classroom or school grounds that adhere to the various solids in the standards, such as blackboard erasers, Kleenex boxes, trashcans, PA speaker boxes, traffic cones, or globes. Having a solid that can be held in the hands activates inherent sense based reasoning and allows students to investigate various orientations and relationships without having to conceptually imagine them.

Space and Dimensions

Students have been considering both two- and three-dimensional geometry since elementary school, but the vast majority of geometric learning prior to and within the Geometry course focuses on two-dimensional space. As students mature and gain experience in the world, they are increasingly able to cognize the character and measures of three-dimensional space and differentiate it from other spaces, but it should also be noted that many students are still building these understandings as they address the geometry standards. These ideas tie back to the foundational geometric concepts of points, lines, and planes. A full understanding of space and dimension cannot be assumed, and must be assessed and discussed in order to support student learning.

- Do students differentiate between one-, two-, and three-dimensional space and how the lower dimensions are included in the higher?
- Do students understand and can they explain that no quantity of figures in one dimension will ever create an object in a higher dimension, as no quantity of points will form a line and no number of lines will give a plane?
- Do they grasp that we live in a three-dimensional world and that any reference to two-dimensional objects in real space is an abstraction, or an imagined cross-section of real space?
- Do they recognize the difference between the measures in each dimension including length (or distance), area, and volume, and can they relate these different measures to each other?

All these questions indicate key concepts that students may or may not grasp, but can cause difficulty if misunderstood. How students think about circles and spheres, triangles and pyramids or cones, and squares and cubes, can also generate good assessment of understanding and misunderstanding of different dimensions. It can also be confusing for students that surface area is a two-dimensional measure that we apply to three-dimensional solids, just as we can measure the length of a curved line even though it travels across a plane.

Finally, it may be useful to consider how students apply their two-dimensional understanding of congruence and similarity to three-dimensional objects. Some students will transfer such concepts fluently and understand that corresponding features in all dimensions must be attended to, while others may focus on congruent features in planes and not attend to aspects of the third dimension.

Conceptual and Procedural Balance

As students solve problems about volume of solids in standard GMD.3, it is important to uphold the balance between conceptual and procedural learning by being intentional about the presentation and use of formulas. Students may use formulas for the volumes of the different solids and even for the surface area of a rectangular prism, and these formulas may be useful, but what is most important is a deep conceptual understanding of why each formula works and how it successfully measures the space in question.

The calculation of volume of a rectangular prism extends the formula for rectangular area by including the new dimension. As long as the student attends to all three dimensions and multiplies each, in any order, they will find the correct volume. This can also be understood as extending the area of the base through the third dimension of the height, or base area times height. This view rests on an understanding of the relationship between two- and three-dimensional space. This can also be applied to the volume of a cylinder, which can be thought of as the area of the circular base being extended through the third dimension of the height (base area times height). The need for meaning applies to finding surface area as well, which includes no new calculations or understandings. But students must apply what they know about rectangular areas and account for all faces, summing the values in any order. If they can find surface area from their understanding and attend to the entire surface, they likely do not require a formula.

It is important that students can explain the meaning of the formulas and why they work. Students that understand how the volume formulas extend what they already know about two-dimensional measures are less likely to see them as something to be memorized without meaning and more likely to derive them when needed based on their own conceptual understanding. Questions about why pyramid and cone volumes include 1/3 and why sphere volume includes 4/3 should be addressed and justified, and can lead to very productive discussions that increase the meanings of volume broadly and the formulas in particular.

Daily Formative Assessment

As students apply their prior understandings about space and dimensions to consider three-dimensional solids, daily assessment of their understandings and misconceptions continues to be essential. Formative assessment questions help students connect procedures to the underlying concepts that justify those procedures, as in the use of volume formulas. This can be seen as the "why" that justifies the "how." Many of the concepts that justify these formulas and procedures are connected to ideas and fluencies regarding two-dimensional space that students do have access to, and teachers can be intentional about helping students make those connections.

Assessing student understanding about three-dimensional space can be difficult, and the temptation to assume is ever-present, but intentional questions always help us to understand student thinking whether they show understanding or misconceptions. Broad assessment questions, like "What is volume?"; "How do you find the volume of a rectangular prism?"; and "What does it mean for a solid to have symmetry?" may be essential questions for a student or group of students as they address these standards.

Essential questions should be intentionally planned and asked repeatedly throughout the learning. Students benefit from answering from their own thinking and refining their expression over time, and teachers benefit from the subtle anecdotal assessment they receive. This type of informal formative assessment shapes ongoing instruction. Teachers who include open conceptual questions as part of daily lesson plans develop more effective questioning practice overall.

Daily formative assessment may be formal or informal. It is often anecdotal and generated from classroom discussions with individual students or groups of students. Formative assessment is based on asking good questions, and can be gained through checks for understanding where students respond in the moment either verbally or physically (raising hands, or giving a thumbs up or thumbs down). It can also be generated through exit tickets or reflection questions given at the end of class or at the completion of an activity.

In all cases, formative assessment is conducted while students are still learning the material. Assessment should generate data about *how* students are learning the material and what misconceptions are arising, so that teachers have time to adjust instruction and students have additional opportunities to build understanding.

Tasks and activities designed for formative assessment are also an essential part of daily formative assessment.

The following links show some examples of formative assessment tasks and lessons from the Mathematics Assessment Project that involve three dimensional solids.

- Calculating Volumes of Compound Objects
- Evaluating Statements about Enlargements

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not mastering the standard. What is it that they do not understand? And how far back do their misconceptions go in the prerequisite standards?

If we can identify the missing concepts, we can begin to understand why they are struggling with the current standard. Students may also begin to grasp some of the conceptual parts of the standards and yet struggle with symbolic representations and notation.

For students who struggle with three-dimensional figures, it is important to identify how they think about space and dimensions in general, and whether their knowledge about two-dimensional geometry includes significant conceptual understanding or rests more on procedural fluency.

It is also important to be attentive to whether student struggles are about the underlying concepts or about the procedural aspects of the standards such as formulas. Observing students working and solving problems with hands-on solids, either individually or with a group, can lead to valuable assessment about misunderstandings.

Upholding a consistent expectation for justification and explanation can also be helpful in assessing areas of student struggle with this content. How students define the objects in question, the operations we apply to those objects, and the underlying concepts at hand, often gives vital evidence that describes their struggle and misconceptions.

In addition to the broad formative assessment questions noted above, the following questions can be helpful in assessing misconceptions and developing understanding and procedural fluency related to three-dimensional figures.

- What are some differences between two and three-dimensional space?
- What are some things that can be difficult when designing a net?
- Is there more than one way to design a net for a specific solid figure? How do you know?
- How do nets connect with the idea of surface area?
- What does it mean for a solid to have symmetry?
- What does it mean for two solids to be congruent?
- What does it mean for two solids to be similar?
- How would you determine that two solids are truly congruent?
- How would you determine that two solids are similar?
- What measures would congruent spheres have in common? Congruent cones?
- What measures define the following solids? (prism, cone, regular pyramid, cylinder, and sphere)
- What is surface area?
- What is volume?

- How are surface and volume different?
- Why does the volume formula for the following solids work? (prism, cone, regular pyramid, cylinder, and sphere)
- What does each part of this volume formula represent?
- Why is the volume of a solid expressed in cubic units?
- What is a design problem? Can you suggest an example?

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including the following:

- Revoicing – "So you're saying that…."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work. Academic vocabulary actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

Encourage Student Sense-making

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are "Why?"; "What does that mean?"; and "How do you know?" These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

In addition to attending to the precise vocabulary around solids, it is important to notice when students use words that indicate confusion about dimension. This could include talking about the volume of a circle, calling a prism a rectangle, or giving a volume in inches. It could also include students referring to a volume as an area or struggling to identify and name the three dimensions of a given solid. Each of these instances could involve misconceptions or simply be cases of imprecise language, but they are all opportunities for that student and all students in the discussion to attend to the important concepts with precise language.[MP.6]

Vocabulary

As noted above, key vocabulary terms need to be discussed and defined by students again and again, allowing them to build and refine their understanding of each concept and the connections between them.

- Three-Dimensional Solid
- Net
- Symmetry
- Congruent Solid
- Similar Solid
- Face
- Vertex
- Edge
- Prism
- Parallel Faces
- Regular Pyramid
- Cylinder
- Cone
- Sphere
- Volume
- Surface Area
- Composite Solid
- Design Problems

References

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* https://www.nctm.org/Standards-and-Positions/Principles-and-Standards/

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. The National Academies Press. https://nap.nationalacademies.org/catalog/9822/adding-it-up-helping-children-learn-mathematics

Resources

GeoGebra for Teaching and Learning Math (https://www.geogebra.org)

Mathematics Assessment Project

Transformations

- Transforming 2D Figures
  https://www.map.mathshell.org/lessons.php?unit=9365&collection=8&redir=1

Triangles

- Circles in Triangles
  https://www.map.mathshell.org/tasks.php?unit=HA08&collection=9&redir=1
- Hopewell Geometry
  https://www.map.mathshell.org/tasks.php?unit=HA05&collection=9&redir=1
- Evaluating Conditions for Congruency
  https://www.map.mathshell.org/lessons.php?unit=9315&collection=8&redir=1
- Deducting Relationships: Floodlight Shadows
  https://www.map.mathshell.org/lessons.php?unit=9305&collection=8&redir=1

Quadrilaterals and Other Polygons

- Floor Pattern
  https://www.map.mathshell.org/tasks.php?unit=HA10&collection=9&redir=1
- Square
  https://www.map.mathshell.org/tasks.php?unit=HA22&collection=9&redir=1

- Evaluating Statements about Length and Area
  https://www.map.mathshell.org/lessons.php?unit=9310&collection=8&redir=1

Circles

- Temple Geometry
  https://www.map.mathshell.org/tasks.php?unit=HA13&collection=9&redir=1
- Circles and Squares
  https://www.map.mathshell.org/tasks.php?unit=HE15&collection=9
- Inscribing and Circumscribing Right Triangles
  https://www.map.mathshell.org/lessons.php?unit=9330&collection=8&redir=1
- Solving Problems with Circles and Triangles
  https://www.map.mathshell.org/lessons.php?unit=9335&collection=8&redir=1
- Circles in Triangles

Three-Dimensional Figures

- Propane Tanks
  https://www.map.mathshell.org/tasks.php?unit=HE16&collection=9&redir=1
- Calculating Volumes of Compound Objects
  https://www.map.mathshell.org/lessons.php?unit=9345&collection=8&redir=1
- Evaluating Statements About Enlargements
  https://www.map.mathshell.org/lessons.php?unit=9320&collection=8&redir=1